6.1 Past and Future

The notion that events proceed in a well defined sequence is unquestionable in classical mechanics. Events occur one after the other, and our knowledge concerning the events at one time allows us to predict what will occur at another time. One can unambiguously determine whether events lie in the past or future of other events. Given two events, A and B, one can compute which event occurred first. It may be, that event A causes event B, in which case, event A must have preceded event B.

In this chapter, we are interested in whether the well defined classical concepts of temporal ordering have a quantum analogue. In other words, given two quantum mechanical systems, can we measure which system attains a particular state first. Can we decide whether an event occurs in the past or future of another event. The problem of measuring the time ordering of two events is in some sense more primitive and fundamental a concept than that of measuring the time of an event.

We saw previously that one cannot measure the time-of-arrival to an accuracy better than $1/\bar{E}_k$ where $\bar{E}_k$ is the kinetic energy of the particle. This leads one to suspect that the ordering of events may not be an unambiguous concept in quantum mechanics. However, for a single quantum event A, although one cannot determine the time an event occurred to arbitrary accuracy, it can be argued that one can often measure whether A occurred before or after a fixed time t_B to any desired precession.

Consider a quantum system initially prepared in a state ψ_A and an event A which corresponds to some projection operator Π_A acting on this state. For example, we could initially prepare an atom in an excited state, and Π_A could represent a projection onto all states where the atom is in its ground state i.e. the atom has decayed. ψ_A could also represent a particle localized in the region $x < 0$ and Π_A could be a projection onto the positive x-axis. In this case, the event A corresponds to the particle arriving to $x = 0$.

If the state evolves irreversible to a state for which $\Pi_A \Psi(t) = 1$, then we can easily measure whether the event A has occurred at any time t. We could therefore measure whether a free particle arrives to a given location before or after a classical time t_B. Of course, for many systems, the system will not irreversible evolve to the required state. For example, a particle influenced by a potential may cross over the origin many times. However, for an event such as atomic decay, the probability of the atom being re-excited is relatively small, and one can argue that the event is more or less irreversible.

For the case of a free particle which is traveling towards the origin from $x < 0$ one can argue that if at a later time I measure the projection operator onto the positive axis and find it there, then the particle must have arrived to the origin at some earlier time. This is in some sense a definition, because we know of no way to measure the particle being at the origin without altering its evolution or being extremely lucky and happening to measure the particle's location when it is at the origin.

While measuring whether an event happened before or after a fixed time t_B may be possible, we will find that for two quantum systems, one cannot in general measure whether the time t_A of event A, occurred before or after the time t_B of event B.

In Section 6.2, confining ourselves to a particular example of order of events, we will consider the question of order of arrival in quantum mechanics. Given two particles, can we determine which particle arrived first to the location x_a. Using a model detector, we find that there is always an inherent inaccuracy in this type of measurement, given by $1/\bar{E}$ where $\bar{E}$ is the typical total energy of the two particles. This seems to suggest that the notion of past and future is not a well defined observable in quantum mechanics.

Note that the measurements we are considering here are continuous measurements, as opposed to the impulsive measurement of an operator. One could, for example, determine the order of arrival by measuring the operator

$$\mathbf{O} = sgn(\mathbf{T}_x - \mathbf{T}_y) \tag{6.189}$$

where T_x and T_y are the time-of-arrival operators associated with each particle.

In Section 6.3 we discuss measurements of coincidence. I.e., can we determine whether both particles arrived at the same time. Such measurements allow us to change the accuracy of the device before each experiment. We find that the measurement fails when the accuracy is made better than $1/\bar{E}$.

In Section 6.4 we discuss the relationship between ordering of events and the resolving power of Heisenberg's microscope, and argue that in general, one cannot prepare a two particle state which is always coincident to within a time of $1/\bar{E}$.

6.2 Which first?

We now examine a case where the time t_B is not given by a classical clock, but rather a quantum system. Consider two free particles (which we will label as x and y) initially localized to the right of the origin, and traveling to the left. We then ask whether one can measure which particle arrives to the origin first. The Hamiltonian for the system and measuring apparatus is given by

$$\mathbf{H} = \frac{\mathbf{P_x}}{2m} + \frac{\mathbf{P_y}}{2m} + \mathbf{H}_i \tag{6.190}$$

where $\mathbf{H}_i$ is some interaction Hamiltonian. For example, a promising interaction Hamiltonian is

$$\mathbf{H}_i = \alpha\delta(\mathbf{x})\theta(-\mathbf{y}) \tag{6.191}$$

with α going to infinity. If the y-particle arrives before the x-particle, then the x-particle will be reflected back. If the y-particle arrives after the x-particle, then neither particle

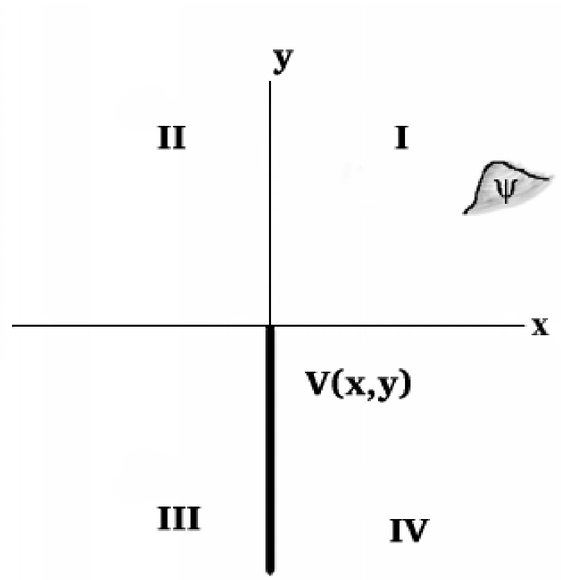

Figure 6.3: A potential which can be used to measure which of two particles came first (given by $V(x, y) = \alpha\delta(\mathbf{x})\theta(-\mathbf{y})$). The wave function for two incoming particles in one dimension looks like a single wave packet in two dimensions travelling towards the origin.

sees the potential, and both particles will continue traveling past the origin. One can therefore wait a sufficiently long period of time, and measure where the two particles are. If both the x and y particles are found past the origin, then we know that the x-particle arrived first. If the y-particle is found past the origin while the x-particle has been reflected back into the positive x-axis then we know that the y-particle arrived first.

Classically, this method would appear to unambiguously measure which of the two particle arrived first. However, in quantum mechanics, this method fails. From (6.190) we can see that the problem of measuring which particle arrives first is equivalent to deciding where a single particle traveling in a plane arrives. Two particles localized to the right of the origin is equivalent to a single particle localized in the first quadrant (see Figure 6.3). The question of which particle arrives first, becomes equivalent to the question of whether the particle crosses the positive x-axis or the positive y-axis.

The Hamiltonian (6.191) is therefore equivalent to the problem of scattering off a thin edge. Classically, particles which do not scatter off the edge will travel to the third quadrant (x arrived first), while particles which scatter off the edge will be found in the fourth quadrant (y arrived first). However, quantum mechanically, we find that sometimes the particle is found in the two classically forbidden regions, I and II. If the particle is found in either of these two regions, then we cannot determine which particle arrived first.

The solution for a plane wave which makes an angle θ_o with the x-axis is well known[46]. If the boundary condition is such that $\psi(r,\theta) = 0$ on the negative y-axis, then the solution is

$$\psi(r,\theta) = \frac{1}{\sqrt{i\pi}} \left\{ e^{-ikr\cos(\theta-\theta_o)} \Phi[\sqrt{2kr}\cos(\frac{\theta-\theta_o}{2})] - e^{ikr\cos(\theta+\theta_o)} \Phi[-\sqrt{2kr}\sin(\frac{\theta+\theta_o}{2})] \right\} \tag{6.192}$$

where $\Phi(z)$ is the error function.

Asymptotically, this solution looks like

$$\psi \simeq \begin{cases} e^{-ikr\cos(\theta-\theta_o)} + f(\theta)\frac{e^{ikr}}{\sqrt{r}} & -\theta_o < \theta < \pi + \theta \\ e^{-ikr\cos(\theta-\theta_o)} - e^{ikr\cos(\theta+\theta_o)} + f(\theta)\frac{e^{ikr}}{\sqrt{r}} & -\theta_o > \theta > -\pi/2 \\ f(\theta)\frac{e^{ikr}}{\sqrt{r}} & \pi - \theta_o < \theta < 3\pi/2 \end{cases} \tag{6.193}$$

where

$$f(\theta) = -\sqrt{\frac{i}{8\pi k}} \left[\frac{1}{\sin(\frac{\theta+\theta_o}{2})} + \frac{1}{\cos(\frac{\theta-\theta_o}{2})} \right], \tag{6.194}$$

where the above approximation is not valid when $\cos(\frac{\theta-\theta_o}{2})$ or $\sin(\frac{\theta+\theta_o}{2})$ is close to zero.

Since we demanded that the particle was initially localized in the first quadrant, the initial wave cannot be an exact plane wave, but we can imagine that it is a plane wave to a good approximation.

We see from the solution above that the particle can be found in the classically forbidden regions of quadrant I and II. For these cases, we cannot determine which particle arrived first. This is due to interference which occurs when the particle is close to the origin (the sharp edge of the potential). The amplitude for being scattered off the region around the edge in the direction θ is given by $|f(r,\theta)|^2$.

It might be argued that since these particles scattered, they must have scattered off the potential, and therefore they represent experiments in which the y-particle arrived first. However, this would clearly over count the cases where the y-particle arrived first. We could have just as easily have placed our potential on the negative x-axis, in which case, we would over count the cases where the x-particle arrived first.

In the "interference region" we cannot have confidence that our measurement worked at all. We should therefore define a "failure cross section" given by

$$\begin{aligned} \sigma_f &= \int_0^{2\pi} |f(\theta)|^2 \\ &= \frac{1}{k\cos(\frac{\theta_o}{2})} \end{aligned} \tag{6.195}$$

From (6.195) we can see that cross section for scattering off the edge is the size of the particle's wavelength multiplied by some angular dependence. Therefore, if the particle arrives within a distance of the origin given by

$$\delta x > 2/k \tag{6.196}$$

the measurement fails. We have dropped the angular dependence from (6.195) – the angular dependence is not of physical importance for measuring which particle came first, as it depends on the details of the potential (boundary conditions) being used. The particular potential we have chosen is not symmetrical in x and y. From this we can conclude that if the particle arrives to within one wavelength of the origin, then there is a high probability that the measurement will fail.

If we want to relate this two-dimensional scattering problem back to two particles traveling in one dimensional, we need to use the relation

$$\delta t \simeq \frac{m\delta x}{k} \tag{6.197}$$

In other words, our measurement procedure relies on making an inference between time measurements and spatial coordinates. The last two equations then give us

$$\delta t > \frac{1}{E} \quad . \tag{6.198}$$

One will not be able to determine which particle arrived first, if they arrive within a time $1/E$ of each other, where E is the total kinetic energy of both particles. Note that Equation (6.198) is valid for a plane wave with definite momentum k. For wave functions for which $dk << k$, one can replace E by the expectation value $\langle E \rangle$. However, for wave functions which have a large spread in momentum, or which have a number of distinct peaks in k, then to ensure that the measurement almost always works, one must measure the order of arrival with an accuracy given by

$$\delta t > \frac{1}{\bar{E}} \tag{6.199}$$

where $\bar{E}$ is the minimum typical total energy [1].

Although it seemed plausible that one could measure which particle arrived first, we found that if the particles are coincident to within $1/\bar{E}$, then the measurement fails.

6.3 Coincidence

In the previous model for measuring which particle arrived first, we found that if the two particles arrived to within $1/\bar{E}$ of each other, the measurement did not succeed. The width $1/\bar{E}$ was an inherent inaccuracy which could not be overcome. However, in our simple model, we were not able to adjust the accuracy of the measurement.

It is therefore instructive to consider a measurement of "coincidence" alone for which one can quite naturally adjust the accuracy of the experiment. Given two particles

[1] For example, one need not be concerned with exponentially small tails in momentum space, since the contribution of this part of the wave function to the probability distribution will be small. If however, $\psi(E)$ has two large peaks at E_{small} and E_{big} spread far apart, then if δt does not satisfy $\delta t > 1/E_{small}$ one will get a distorted probability distribution. For a discussion of this, see Chapter 3.

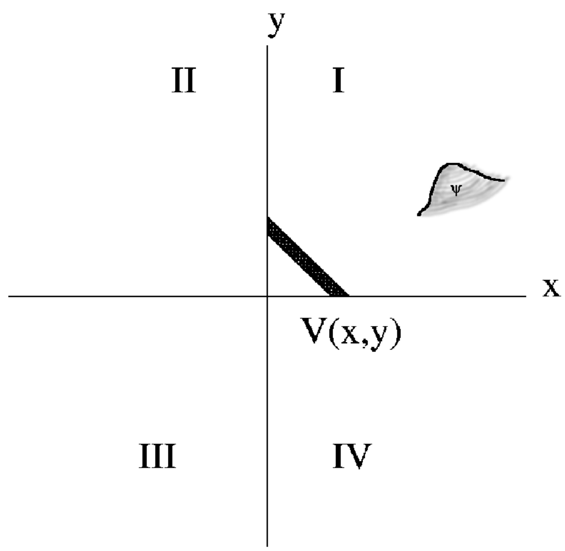

Figure 6.4: Potential for measuring whether two particles are coincident.

traveling towards the origin, we ask whether they arrive within a time δt_c of each other. If the particles do not arrive coincidently, then we do not concern ourselves with which arrived first. The parameter δt_c can be adjusted, depending on how accurate we want our coincident "sieve" to be. We will once again find that one cannot decrease δt_c below $1/\bar{E}$.

A simple model for a coincidence measuring device can be constructed in a manner similar to (6.191). Mapping the problem of two particles to a single particle in two dimensions, we could consider an infinite potential strip of length $2a$ and infinitesimal thickness, placed at an angle of $\pi/4$ to the x and y axis in the first quadrant (see Figure 6.4). Particles which miss the strip, and travel into the third quadrant are not coincident, while particles which bounce back off the strip into the first quadrant are measured to be coincident. I.e. if the x-particle is located within a distance a of the origin when the y-particle arrives (or visa versa), then we call the state coincident.

Classically, one expects there to be a sharp shadow behind the strip. Quantum mechanically, we once again find an interference region around the strip which scatters particles into the classically forbidden regions of quadrant two and four. The shadow is not sharp, and we are not always certain whether the particles were coincident.

A solution to plane waves scattering off a narrow strip is well known and can be found in many quantum mechanical texts (see for example [46] where the scattered wave is written as a sum of products of Hermite polynomials and Mathieu functions). However, for our purposes, we will find it convenient to consider a simpler model for measuring coincidence, namely, an infinite circular potential of radius a, centered at the origin.

$$H_i = \alpha V(r/a) \tag{6.200}$$

where $V(x)$ is the unit disk, and we take the limit $\alpha \to \infty$.

It is well known that if $a < 1/k$, then there will not be a well-defined shadow behind the disk. To see this, consider a plane wave coming in from negative x-infinity. It can be expanded in terms of the Bessel function $J_m(kr)$ and then written asymptotically ($r \gg 1$) as a sum of incoming and outgoing circular waves.

$$\begin{aligned} e^{ikx} &= \sum_{m=0}^{\infty} \epsilon_m i^m J_m(kr) \cos m\theta \\ &\simeq \sqrt{\frac{1}{2\pi ikr}} \left[e^{ikr} \sum_{m=0}^{\infty} \epsilon_m \cos m\theta + ie^{-ikr} \sum_{m=0}^{\infty} \epsilon_m \cos m(\theta - \pi) \right] \quad . \end{aligned} \tag{6.201}$$

where ϵ_m is the Neumann factor which is equal to 1 for $m = 0$ and equal to 2 otherwise.

Since it can be shown that

$$\sum_{m=0}^{M} \epsilon_m \cos m\theta = \frac{\sin{(M + \frac{1}{2})\theta}}{\sin \frac{1}{2}\theta} \tag{6.202}$$

The two infinite sums approach $2\pi\delta(\theta)$ and $2\pi\delta(\theta - \pi)$ respectively, and so the incoming wave comes in from the left, and the outgoing wave goes out to the right. The presence of the potential modifies the wave function and in addition to the plane wave, produces a scattered wave

$$\psi = e^{ikx} + \frac{e^{ikr}}{\sqrt{r}} f(r\theta) \tag{6.203}$$

where

$$\frac{e^{ikr}}{\sqrt{r}} f(r,\theta) = -i \sum_{m=0}^{\infty} \epsilon_m e^{\frac{1}{2}m\pi i - i\delta_m} \sin\delta_m H_m(kr) \cos m\theta \quad , \tag{6.204}$$

$H_m(kr)$ are Hermite polynomials and

$$\tan\delta_m = \frac{-J_m(ka)}{N_m(ka)} \tag{6.205}$$

($N_m(ka)$ are Bessel functions of the second kind). For large values of r, the wave function can be written in a manner similar to (6.201), except that the outgoing wave is modified by the phase shifts δ_m.

$$\psi \simeq \frac{1}{\sqrt{2\pi i k}} i \sum_{m=0}^{\infty} \epsilon_m \cos m(\theta - \pi) \frac{e^{-ikr}}{\sqrt{r}} + \frac{e^{ikr}}{\sqrt{r}} f(r,\theta) \quad . \tag{6.206}$$

where

$$f(r,\theta) \simeq \frac{1}{\sqrt{2\pi i k}} \sum_{m=0}^{\infty} \epsilon_m e^{-2i\delta_m(ka)} \cos m\theta \tag{6.207}$$

In the limit that $ka \gg m$ the phase shifts can be written as

$$\delta_m \simeq ka - \frac{\pi}{2}(m + \frac{1}{2}) \quad . \tag{6.208}$$

In the limit of extrememely large a (but $a < r$), the outgoing waves then behave as

$$f(r,\theta) \simeq \lim_{M\to\infty} -i \frac{1}{\sqrt{2\pi i k}} e^{-2ika} \frac{\sin\,(M + \frac{1}{2})(\theta - \pi)}{\sin\frac{1}{2}(\theta - \pi)} \tag{6.209}$$

where once again we see that the angular distribution goes as the delta function $\delta(\theta - \pi)$. The disk scatters the plane wave directly back, and a sharp shadow is produced. We see therefore, that in the limit of $ka \gg 1$, our measurement of coincidence works.

The differential cross section can in general be written as

$$\begin{aligned} \sigma &= |f(\theta)|^2 \\ &= |\sum_{m=0}^{\infty} \epsilon_m e^{-2i\delta_m(ka)} \cos m\theta|^2 \end{aligned} \tag{6.210}$$

For $ka \gg 1$ (but still finite), (6.210) can be computed using our expression for the phase shifts from (6.208), and is given by

$$\sigma(\theta) \simeq \frac{a}{2}\sin\frac{\theta}{2} + \frac{1}{2\pi k}\cot^2\frac{\theta}{2}\sin^2 ka\theta \tag{6.211}$$

The first term represents the part of the plane wave which is scattered back, while the second term is a forward scattered wave which actually interferes with the plane-wave. The reason it appears in our expression for the scattering cross section is because we have written our wave function as the sum of a plane-wave and a scattered wave, and so part of the scattered wave must interfere with the plane-wave to produce the shadow behind the disk.

For $ka \ll m$, the phase shifts look like

$$\delta_m(ka) \simeq \frac{\pi m}{(m!)^2}\left(\frac{ka}{2}\right)^{2m} \qquad m \neq 0 \tag{6.212}$$

and

$$\tan\delta_0(ka) \simeq \frac{-\pi}{2\ln ka} \tag{6.213}$$

As a result, for $ka \ll 1$, δ_0 is much greater than all the other δ_m and the outgoing solution is almost a pure isotropic s-wave.

For $ka \ll 1$ the only contribution to (6.210) comes from δ_0 and the differential cross section becomes

$$\sigma(\theta) \simeq \frac{\pi}{2k\ln^2 ka} \tag{6.214}$$

and is isotropic. In other words, no shadow is formed at all, and particles are scattered into classically forbidden regions. We see therefore, that as long as the s-wave is dominant, our measurement fails. The s-wave will cease being dominant when δ_0 is of the same order as δ_1. As can be seen from Equation 6.208, δ_1/δ_0 approaches a limiting value of 1 when a sharp shadow is produced. It is only when $\delta_1/\delta_0 \simeq 1$ that the cross-section no longer

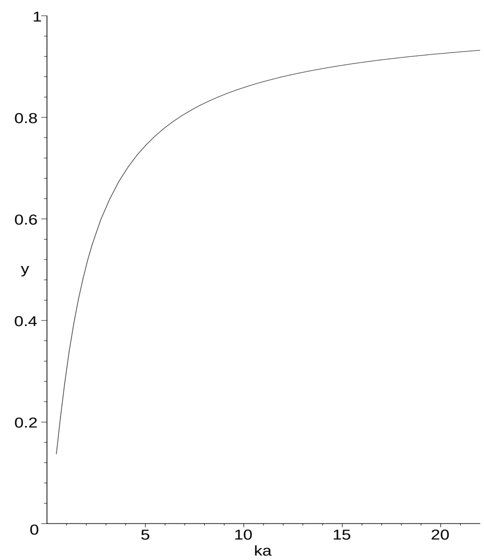

Figure 6.5: Phase shifts for coincidence detector ($\delta_1(ka)/\delta_0(ka)$ vs. ka)

depends on k. This is what we require then, for the probability of our measurement to succeed independently of the energy of the incoming particles. From a plot of δ_1/δ_0 we see that this only occurs when $ka \gg 1$ (Figure 6.5). Our condition for an accurate measurement is therefore that $a \gg 1/k$. Since $\delta t_c \simeq am/k$ we find

$$\delta t_c \gg 1/E \tag{6.215}$$

6.4 Coincident States

We have seen that we can only measure coincidence to an accuracy of $\delta t_c = 1/\bar{E}$. We shall now show that one cannot prepare a two particle system in a state ψ_c which always arrives coincidentally within a time less than δt_c. In other words, one cannot prepare a system in a state which arrives coincidentally to greater accuracy than that set by the limitation on coincidence measurements.

Preparing a state ψ_c corresponds to preparing a single particle in two dimensions which always arrives inside a region $\delta r = p\delta t_c/m$ of the origin. In other words, suppose we were to set up a detector of size δr at the origin. If a state ψ_c exists, then it would always trigger the detector at some later time.

Our definition of coincidence requires that the state ψ_c not be a state where one particle arrives at a time $t > \delta t_c$ before the other particle. In other words, if instead, we were to perform a measurement on ψ_c to determine whether particle x arrived at least δt_c before particle y, then we must get a negative result for this measurement.

This latter measurement would correspond to the two-dimensional experiment of placing a series of detectors on the positive y-axis, and measuring whether any of them are triggered by ψ_c. If ψ_c is truly a coincident state, then none of the detectors which are placed at a distance greater than $y = \delta r$ can be triggered. One could even consider a single detector, placed for example, at $(0, \delta r)$, and one would require that ψ_c not trigger this detector.

Now consider the following experiment. We have a particle detector which is either placed at the origin, or at $(0, \delta r)$ (we are not told which). Then after a sufficient length of time, we observe whether it has been triggered. If we can prepare a coincident state ψ_c, then it will always trigger the detector when the detector is at the origin, but never trigger the detector when the detector is at $(0, \delta r)$. This will allow us to determine whether the detector was placed at the origin, or at $(0, \delta r)$. For example, if we use the detectors described in Section 6.3 (namely, just a scattering potential), then some of the time, the particle will be scattered, and some of the time it won't be, and if it is scattered, we can conclude that the potential was centered around the origin rather than around $(0, \delta r)$.

However, as we know from Heisenberg's microscope, a particle cannot be used to resolve anything greater than it's wavelength. In other words ψ_c cannot be used to determine whether the detector is at the origin, or at $(0, \delta r)$ if $\delta r < 2\pi/k$. As a result, ψ_c can only be coincident to a region around the origin of radius less than δr or, coincident within a time $\delta t_c \sim 1/E$.

It is also interesting to consider the situation where we have an event B which must be preceded by an event A. For example, B could be caused by A, or the dynamics could be such that B can only occur when the system is in the state A. One can then attempt to force B to occur as close to the occurrence of event A as possible. This problem has been studied in relation to the maximum speed of quantum computers [39] and it was found that one cannot force the system to evolve at a rate greater than the average energy.

6.5 In Which Direction Does the Light Cone Point

We have argued that we cannot measure the order of arrival for two free particles, if they arrive within a time of $1/\bar{E}$ of each other, where $\bar{E}$ is their typical total kinetic energy. If we try to measure whether they arrive within a time δt_c of each other, then our measurement fails unless we have at least $\delta t_c > 1/\bar{E}$. Furthermore, we cannot construct a two particle state where both particles arrive to a certain point within a time of $1/\bar{E}$ of each other. The results in this Chapter support the limitations we have found for measurements of arrival time.

In general, it appears that for two quantum mechanical events, one cannot determine which one precedes the other to arbitrary accuracy. Determining what order events occur in is not a trivial problem. This is interesting in light of attempts to construct a quantum theory of gravity. In classical general relativity, the metric allows you to determine whether two space-time events are space-like separated or time-like separated, and determine, relative to a coordinate system, which event occurred first. However, when quantum mechanics is taken into account, it does not appear that there is a way to do this to arbitrary accuracy, without affecting the system. In some sense, one may not be able to determine which way the light cone points.

One could use the arrival of arbitrarily energetic particles in order to denote space-time events, and although one can increase the energy of the particles in order to increase the accuracy with which one is able to measure the order of events, at some point the energy of the particles will effect the curvature of the neighboring space time.