

### 5.1 A Limitation on Traversal Time Measurements

In Chapter 3, we considered various clock models for measuring the time it takes for a free particle to arrive to a given location  $x_A$ . Because the energy of the clock increases with its precision, we argued that the accuracy of a time-of-arrival detector cannot be greater than  $1/E_k$ , where  $E_k$  is the kinetic energy of the particle. Measurements of traversal time [16] are analogous to that of time-of-arrival. One tries to measure how long it takes a particle to travel between two fixed locations  $x_1$  and  $x_2$ . Although no proof has yet been found for the restriction on time-of-arrival accuracy, in this Chapter we provide model independent arguments that a necessary minimum inaccuracy on traversal time measurements is given by

$$\delta T_F > 1/E_p. \quad (5.143)$$

We do this by arguing that a traversal time measurement is also a simultaneous measurement of position and momentum, and that (5.143) is required in order to preserve the Heisenberg uncertainty relationship. Once again (5.143) is not analogous to the Heisenberg Energy-time uncertainty relationship. It reflects the inherent inaccuracy of every individual measurement, while the Heisenberg uncertainty relationships refer to well-defined and perfectly accurate measurements made on ensembles.

This chapter proceeds as follows. In section 5.2 we motivate the notion that traversal time is a measurement of momentum by looking at measuring the traversal-distance. In section 5.3 we discuss a physical model for measuring the traversal time, and show the relation between (5.143) and the uncertainty principle. The main result of this Chapter is given in Section 5.4, where we provide a model independent argument for (5.143), as well as a qualitative proof.

## 5.2 Measuring Momentum Through Traversal-Distance

The measurement of traversal-distance may be considered the space-time *dual* of the measurement of traversal time: instead of fixing  $x_1$  and  $x_2$  and measuring  $t_F = t_2 - t_1$ , one fixes  $t_1$  and  $t_2$  and measures  $x_F = x_2 - x_1$ . It is instructive to examine first this simpler case of traversal-distance and point out the similarities and the differences.

Unlike the case of traversal time, a measurement of traversal-distance can be described by the standard von Neumann interaction. For a free particle the Hamiltonian is

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \mathbf{Q}\mathbf{x} \left[ \delta(t - t_1) - \delta(t - t_2) \right] \quad (5.144)$$

where  $\mathbf{Q}$  is the coordinate conjugate to the pointer variable  $\mathbf{P}$ . The change in  $\mathbf{P}$  yields the traversal-distance:

$$\mathbf{P}(t > t_2) - \mathbf{P}_0 = \mathbf{x}(t_2) - \mathbf{x}(t_1) = \mathbf{x}_F. \quad (5.145)$$

However the measurement of the traversal-distance provides additional information: it also determines the momentum  $\mathbf{p}$  of the particle *during* the time interval  $t_1 < t < t_2$ . From the equations of motion we get:

$$\mathbf{p}(t) = \begin{cases} \mathbf{p}_o, & t < t_1 \text{ or } t > t_2 \\ \mathbf{p}_o - \mathbf{Q}, & t_1 < t < t_2 \end{cases} \quad (5.146)$$

and

$$\mathbf{x}(t) = \begin{cases} x_0 + \frac{\mathbf{p}_o}{m}t, & t \leq t_1 \\ x_0 + \frac{\mathbf{p}_o}{m}t_1 + \frac{\mathbf{p}_o - \mathbf{Q}}{m}(t - t_1), & t_1 \leq t \leq t_2 \end{cases} \quad (5.147)$$

and therefore,

$$m \frac{\mathbf{P}(t > t_2) - \mathbf{P}_0}{t_2 - t_1} = \mathbf{p}_0 - \mathbf{Q} = \mathbf{p}(t_1 \leq t \leq t_2). \quad (5.148)$$

Thus, one can determine simultaneously and to arbitrary accuracy the traversal-distance and the momentum in intermediate times. This, of course, does not contradict the

uncertainty relations, because  $\mathbf{p}$  commutes with  $\mathbf{x}_F$ , and  $\mathbf{x}$  remains completely uncertain. Similarly, in the case of the traversal time we shall see that the measurement determines also the momentum during the traversal, however unlike the present case, since the particle has to be in the interval  $x_2 - x_1$  during the traversal, it is also a measurement of the location. This indicates that, in the latter case, in order not to violate the kinematic uncertainty principle for  $x$  and  $p$ , the accuracy with which the traversal time,  $T_F$ , or the momentum may be measured must be limited.

### 5.3 Measuring Traversal Time

For traversal time the classical equations of motion suggest that a traversal time operator might be given by

$$\mathbf{T}_F = \frac{mL}{\mathbf{p}}, \quad (5.149)$$

where  $L = x_2 - x_1$ . This operator is self-adjoint, but like the time-of-arrival operator, we shall see that different outcomes are found in a direct measurement of  $T_F$  and a measurement of the operator  $\mathbf{T}_F$ . Furthermore, one can measure the quantity  $\mathbf{T}_F$  at any time, so there is no reason to believe that the particle actually traveled between the two points in the time  $t_F$ . Since  $\mathbf{T}_F$  is only a function of  $\mathbf{p}$ , the measurement will result in the particle's position being spread over all space, so there is no finite amount of time one could wait before being certain that the particle went between the two fixed points. For example, after the measurement of  $\mathbf{T}_F$ , the potential between  $x_1$  and  $x_2$  might change. General traversal time operators would require that one knows the Hamiltonian not only in the past, but also in the future. If one measures the traversal time operator above, then one has to have faith that the Hamiltonian will not change after the time of the measurement  $t_o$  to  $t \rightarrow \infty$ .

It is also commonly accepted that the dwell time operator [31], given by

$$\tau_{\mathbf{D}} = \int_0^\infty dt \mathbf{\Pi}_{x_A}(t) \quad (5.150)$$

where

$$\mathbf{\Pi}_{x_A}(0) = \int_{x_1}^{x_2} |x\rangle \langle x| \quad (5.151)$$

can be used to compute the traversal time<sup>1</sup>. Such a quantity however, cannot be measured, since, as we saw in Chapter 2, the operator  $\mathbf{\Pi}_{x_A}(t)$  does not commute with itself at different times.

$$[\mathbf{\Pi}_{x_A}(t), \mathbf{\Pi}_{x_A}(t')] \neq 0. \quad (5.152)$$

Therefore, one must measure the traversal time in a more physical way. One must demand that if we measure the traversal time to be  $t_F$ , then the particle must actually traverse the distance between  $x_1$  and  $x_2$  in the time given by the traversal time measurement. For example, one could have a clock which runs when the particle is between  $x_1$  and  $x_2$  given by the Hamiltonian [16][32]

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x})\mathbf{Q} \quad (5.153)$$

where the traversal time is given by the variable  $\mathbf{P}$  conjugate to  $\mathbf{Q}$  and the potential  $V$  is equal to 1 when  $x_1 \leq x \leq x_2$  and 0 everywhere else. In the Heisenberg picture, the equations of motion are

$$\dot{\mathbf{x}} = \mathbf{p}/m, \quad \dot{\mathbf{p}} = -\mathbf{Q}(\delta(\mathbf{x} - x_1) - \delta(\mathbf{x} - x_2)) \quad (5.154)$$

$$\dot{\mathbf{P}} = V(\mathbf{x}), \quad \dot{\mathbf{Q}} = 0. \quad (5.155)$$

The particle's momentum is disturbed during the measurement

$$\mathbf{p}' = \sqrt{\mathbf{p}^2 - 2m\mathbf{Q}} \quad (5.156)$$

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<sup>1</sup>in our case, where there is no potential barrier, the dwell time and traversal time are equivalent

where  $\mathbf{p}'$  is the particle's momentum during the measurement, and  $\mathbf{p}$  is the undisturbed momentum. However if the interaction is weak  $Q \ll E_p$ , then after a sufficient time, the clock will read the undisturbed traversal time

$$\begin{aligned} \mathbf{P}(t \rightarrow \infty) - \mathbf{P}(0) &\simeq \int_0^\infty V \left( \mathbf{x}(0) - \frac{\mathbf{p}_o t}{m} \right) dt \\ &= \frac{m(x_2 - x_1)}{\mathbf{p}} \end{aligned} \quad (5.157)$$

If we require an accurate measurement of the traversal time, then a small  $dP$  will result in large values of the coupling  $Q$ . If  $Q$  is too large, the clock can reflect the particle at  $x_1$  and one will obtain a traversal time equal to 0. This therefore imposes a restriction on the accuracy with which one can measure the traversal time. As in Chapter 3 we find that

$$\delta T_F > 1/E_p \quad (5.158)$$

is required in order to be able to measure the traversal time, and

$$\delta T_F \gg 1/E_p \quad (5.159)$$

in order to measure the undisturbed value of the traversal time.

Let us show that the above conditions are consistent with the uncertainty relations for the position and momentum. If (5.159) is satisfied, we have  $Q \ll E$ , and by eq. (5.156) the momentum during the measurement is

$$\mathbf{p}' \simeq \mathbf{p} - \frac{m}{\mathbf{p}} \mathbf{Q}. \quad (5.160)$$

Thus during the measurement, the momentum will be uncertain by an amount

$$dp' \simeq \frac{m}{p_o} dQ. \quad (5.161)$$

In order to know whether the particle entered our detector, we need to be able to distinguish between the case where the pointer is at its initial position  $P = 0$ , and the case

where the particle has gone through the detector  $P = t_F = \frac{mL}{p_o}$ . We therefore need the condition

$$dP < \frac{mL}{p_o}. \quad (5.162)$$

Since at best we have  $dP = 1/dQ$ , we find

$$dp'dx = dp'L > 1. \quad (5.163)$$

The uncertainty relation (5.158) only applies to this particular model clock - it might be possible to accurately measure the traversal time in some clever way. In the following section we will argue that this cannot be done, by demonstrating that this uncertainty applies to all measurements of traversal time.

Finally, we should note that a traversal time detector could be made by measuring the time-of-arrival to  $x_1$  and the time-of-arrival to  $x_2$ . This would require two time-of-arrival clocks each with its own inaccuracy, whereas the model above only has one clock.

#### 5.4 General Argument for a Minimum Inaccuracy

We now consider general measurements of traversal time. We will however, impose some fairly unrestrictive requirements on the measurement. We will assume that the measurement does not prevent the particle from actually traversing the distance between  $x_1$  and  $x_2$ . I.e. we want to be able to say that the particle did indeed traverse the distance  $L$  - otherwise, it is unclear what it is that we are measuring.

We also demand that the result of the measurement corresponds in some sense with the classical notion of traversal time. I.e., we are measuring something like  $mL/p$ . Our measuring device will consist of a pointer, which is set to some initial position  $P_o$  with an uncertainty in the initial value of the pointer of  $dP$ . At the end of the measurement, we assume that the value of the traversal time is inferred accurately by reading the final

value of the pointer. The measured traversal time is then proportional to  $P_f - P_i$ . The relative accuracy of the traversal time will then be given by  $\delta T_f / T_f - dP / (P_f - P_i)$

Another condition we will impose is that the inaccuracy of the measurement,  $\delta T_F$ , is less than the quantity we are trying to measure  $\mathbf{T}_F$  (i.e. we are looking at accurate measurements). Finally, we assume that the experimentalist has no knowledge of the state of the particle, and thus must set the initial state of the measuring device (and its inaccuracy  $\delta P$ ) with no prior knowledge of the ensemble.

Before proceeding with the argument, we should be clear to distinguish between different types of uncertainties. For an operator  $\mathbf{A}$ , there exists a kinematic uncertainty which we will denote by  $d\mathbf{A}$  given by

$$d\mathbf{A}^2 = \langle \mathbf{A}^2 \rangle - \langle \mathbf{A} \rangle^2. \quad (5.164)$$

This is the uncertainty in the distribution of the observable  $A$ . There is also the inherent *inaccuracy* in the measuring device. This is the relevant quantity in equations (5.143) and (5.158). It refers to the uncertainty in the initial state of the measuring device's pointer position  $P$ , and we will denote it by  $\delta A$ . For our measuring devices we have

$$\delta A = dP_o \quad (5.165)$$

This inaccuracy applies to each individual measurement. Lastly, there is the uncertainty  $\Delta A$  which applies to the spread in measurements made on the ensemble. Given a set  $A_M$  of experiments  $i = 1, 2, 3, \dots$  which yield results  $A_i$ , we have

$$\Delta A^2 = \langle A_M^2 \rangle - \langle A_M \rangle^2. \quad (5.166)$$

This uncertainty includes a component due to the kinematic uncertainty of the attribute of the system, and also the inaccuracy of the device. For our measuring device, the kinematic spread in the pointer position at the end of each experiment gives  $\Delta A$

$$\Delta A = dP_f \quad (5.167)$$

The Heisenberg uncertainty relationship  $dA dB > 1$  applies to measurements on ensembles. Given an ensemble, we measure **A** on half the ensemble and **B** on the other half. The uncertainty relation also applies to simultaneous measurements<sup>2</sup>. If we measure **A** and **B** simultaneously on each system in the ensemble, then the distributions of **A** and **B** must still satisfy the uncertainty relationship.

Returning now to the traversal time, we see that it can be interpreted as a simultaneous measurement of position and momentum. We know the particle's momentum  $p$  during the time that it was between  $x = x_1$  and  $x = x_2$  from the classical equations of motion

$$t_F = \frac{mL}{p}. \quad (5.168)$$

In other words, eigenstates of momentum must have traversal times given by equation (5.168). This measurement of momentum is analogous to the measurement described in Section 5.2. Instead of measuring the change in position at two specified times  $t_1$  and  $t_2$ , we are now measuring the difference in arrival times after specifying two different positions  $x_1$  and  $x_2$ . During the measurement, we also know that the particle is somewhere between  $x = x_1$ , and  $x = x_2$ . i.e.. we know that  $x = \frac{x_1+x_2}{2} \pm L/2$ .

The uncertainty relationship also applies to these measured quantities  $\Delta x \Delta p > 1$ . This essentially means that a detector of size  $L$  will disturb the momentum by at least  $2/L$ , so that repeated measurements on an ensemble will give  $\Delta p > 2/L$ . The position of the detector **X** commutes with the momentum of the particle **p** [10] however, we demand that the particle actually travel the distance  $L$ . The particle must actually be inside the detector during the measurement. As a result, **X** must be coupled to the position **x** of

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<sup>2</sup>For a discussion of how the uncertainty relation applies to simultaneous measurements, see for example, Arthurs and Kelly[38] They propose a model for simultaneous measurements using a Hamiltonian  $\mathbf{H} = \delta(t)(\mathbf{P}_1 \mathbf{A} + \mathbf{P}_2 \mathbf{B})$  which measures the variables **A** and **B** using two measurement pointers **Q**<sub>1</sub> and **Q**<sub>2</sub> which are conjugate to **P**<sub>1</sub> and **P**<sub>2</sub>. They show how the Heisenberg uncertainty relation applies to the uncertainty in the outcomes of the measurement of **A** and **B**.

the particle and so a measurement of  $\mathbf{X}$  is also a measurement of  $\mathbf{x}$ . This is what we mean by a local interaction.

We now show why we expect (5.143) to be true. During the measurement of traversal time, the momentum will be disturbed by an amount

$$dp > 2/L. \quad (5.169)$$

If this disturbance is small, then from (5.168) we expect this will cause an inaccuracy given by

$$\begin{aligned} \delta T_F &= \frac{mL}{p^2} dp \\ &> 1/E_p \end{aligned} \quad (5.170)$$

For measurements where the disturbance to the system is minimal (i.e.. when  $dp$  and  $\delta T$  are small) we see that (5.143) gives the minimum inaccuracy on traversal time measurements.

We now proceed with the slightly more rigorous argument. We imagine a traversal time detector which has an inaccuracy given by  $\delta T_F$ . Measurements can then be carried out on arbitrary ensembles with arbitrary Hamiltonians. We will show that by choosing this ensemble appropriately, the uncertainty relationship  $\Delta x \Delta p > 1$  can be violated, unless the traversal time obeys the relationship given by (5.143).

We assume that initially, the pointer on our traversal time detector is given by

$$P_o = \epsilon \quad (5.171)$$

where  $\epsilon$  is a small random number which arises because of the initial inaccuracy of the clock. i.e.. the distribution of  $\epsilon$  is such that  $\langle \epsilon \rangle = 0$  and the clock's initial inaccuracy in pointer position is  $dP_o^2 = \langle \epsilon^2 \rangle$ . It is important to note that this inaccuracy is fixed as an initial condition before any measurements are made. It is a property of the device,

and does not depend on the nature of the ensemble upon which we will be making measurements. For a free Hamiltonian, a measurement of the traversal time will result in a final pointer position given by

$$P_f = P_o + \frac{mL}{p} \quad (5.172)$$

where  $p$  is the momentum of the particle in the absence of any measuring device. For eigenstates of  $\mathbf{p}$  (or states peaked highly in  $p$ ), we demand that the traversal time be given by the classically expected value<sup>3</sup>. Recall that the kinematic spread in the particle's momentum is given by  $d\mathbf{p}^2 = \langle \mathbf{p}^2 \rangle - \langle \mathbf{p} \rangle^2$ . A measurement of the traversal time for a particular experiment  $i$  can take on the values

$$\begin{aligned} T_i &= P_f \\ &= \frac{mL}{p} + \epsilon \end{aligned} \quad (5.173)$$

A given measurement  $T_i$  will allow us to infer the momentum of the particle  $p_i$  during the measurement

$$p_i(T_i) = \frac{mL}{T_i} = \frac{mLp}{mL + p\epsilon}. \quad (5.174)$$

The average value of any power  $\alpha$  of the measured momentum is

$$\langle p_M^\alpha \rangle = \int \left( \frac{mLp}{mL + p\epsilon} \right)^\alpha f(p)g(\epsilon)dpd\epsilon \quad (5.175)$$

where  $f(p)$  gives the distribution of the particle's momentum and  $g(\epsilon)$  is the distribution of the fluctuations. We now choose the mass  $m$  of the ensemble so that we always have

$$\epsilon p \ll mL. \quad (5.176)$$

In other words, we consider measurements on ensembles where the measurement is much more accurate than the quantity being measured. i.e.,  $\delta T_F \ll T_F$ . Indeed for the

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<sup>3</sup>It is possible to include small deviations from the classical value, by including an additional term in (5.172). These fluctuations need to average to zero in order to satisfy the correspondence principle. For small fluctuations, the following discussion is not altered.

example given in the previous section, for every given  $\epsilon$  and  $p$ , we can increase  $E_p$  by choosing a sufficiently large  $m$ , and reach this limit. This limit ensures that  $\langle p_M \rangle$  never diverges, and simplifies our calculation by allowing us to write

$$\langle p_M^\alpha \rangle \simeq \int \left( p - \frac{\epsilon p^2}{mL} \right)^\alpha f(p)g(\epsilon)dpd\epsilon \quad (5.177)$$

For  $\alpha = 1$  we find

$$\begin{aligned} \langle p_M \rangle &\simeq \langle p \rangle - \frac{\langle \epsilon \rangle \langle p^2 \rangle}{mL} \\ &= \langle p \rangle. \end{aligned} \quad (5.178)$$

For  $\alpha = 2$  we find

$$\langle p_M^2 \rangle \simeq \int \left( p^2 - 2\frac{\epsilon p^3}{mL} + \left( \frac{\epsilon p^2}{mL} \right)^2 \right) f(p)g(\epsilon)dpd\epsilon \quad (5.179)$$

$$= \langle p^2 \rangle + \frac{\langle p^4 \rangle \langle \epsilon^2 \rangle}{(mL)^2}. \quad (5.180)$$

This gives us

$$\begin{aligned} \Delta p^2 &= \langle p_M^2 \rangle - \langle p_M \rangle^2 \\ &= \frac{\langle p^4 \rangle \delta T_F^2}{(mL)^2} + dp^2 \end{aligned} \quad (5.181)$$

Since

$$(dE)^2 = \frac{\langle p^4 \rangle}{4m^2} - \langle E \rangle^2 \quad (5.182)$$

we find

$$\Delta p^2 = \left( \frac{2\delta T_F}{L} \right)^2 (dE^2 + \langle E \rangle^2) + dp^2. \quad (5.183)$$

Finally, we arrive at the relation

$$(\Delta x \Delta p)^2 = \delta T_F^2 (\langle E \rangle^2 + dE^2) + \frac{L^2}{4} dp^2. \quad (5.184)$$

The uncertainty relation

$$\Delta x \Delta p > 1 \quad (5.185)$$

then implies

$$\delta T_F^2 > \frac{1 - \frac{1}{4}L^2 dp^2}{\langle E \rangle^2 + dE^2}. \quad (5.186)$$

Now we note that we can arrange our experiment with  $Ldp$  arbitrarily small, by choosing  $dp$  of the ensemble arbitrarily small. i.e.. the uncertainty in the traversal time is small. As a result, in order to ensure that Heisenberg's uncertainty relation is never violated, we must have

$$\delta T_F > \frac{1}{\sqrt{\langle E \rangle^2 + dE^2}}. \quad (5.187)$$

Since  $dp$  is small, we can write

$$\delta T_F > \frac{1}{\langle E \rangle}. \quad (5.188)$$

It is interesting to note that since the momentum operator commutes with the free Hamiltonian, the restriction on traversal time measurements only comes from the dynamical considerations given above.

## 5.5 From Traversal Time to Barrier Tunneling Time

We have seen that the measurement of the traversal time given two positions cannot be made arbitrarily accurate. We have argued this by looking at a simple model for measuring traversal time, and we have also given a model independent, qualitative proof of this which applies when the measurement only disturbs the system slightly. Finally we have given a more rigorous argument which applies when the uncertainty in traversal time is small. This strongly suggests that the limitation on measurements of arrival times is a general rule and not just an artifact of the types of models considered so far. Operators for both the traversal time and the arrival time don't seem to correspond to physical (continuous) processes. The case of traversal time is different from time-of-arrival in that there does exist a self-adjoint traversal time operator, and the semi-bounded spectrum of the Hamiltonian does not seem to play an important role in the

restriction on measurement accuracy. The accuracy restriction on traversal time may be particularly important for experiments on barrier tunneling time. If one uses a physical clock to measure the time it takes for a particle to travel from one location to another, with a barrier situated somewhere between the two locations [17][32], then the accuracy of this clock may affect the tunneling particle. The limitation presented in this Chapter seem to indicate that measurements of barrier tunneling times would also need to be inherently inaccurate, because if one tries to measure the tunneling time too accurately, the particle may be unable to tunnel. Our result concerning traversal time indicates that the barrier tunneling time also cannot be precisely defined.