4.1 Indirect Time-of-Arrival Measurements

In the previous chapter, we saw that one cannot measure the time-of-arrival of a free particle to arbitrary accuracy by coupling the particle to a clock. Still, one can imagine an indirect determination of arrival time by a measurement of some regularized time-of-arrival operator $\mathbf{T}(\mathbf{x}(t), \mathbf{p}(t), x_A)$ [9]. In quantum mechanics, ordinary observables like position and momentum *are* represented by operators at a fixed time t. However, we will show that there is no operator associated with the time it takes for a particle to arrive to a fixed location. In Section 4.2 we will prove formally that in general a Hermitian time-of-arrival operator with a continuous spectrum can only exist for systems with an unbounded Hamiltonian. This is because the existence of a time-of-arrival operator requires the existence of a time operator which is conjugate to the Hamiltonian. As is argued in Section 4.3, since $\mathbf{T}$ can be measured with arbitrary accuracy it does not correspond to the result obtained by the direct measurement discussed in Chapter 3 .

In Section 4.4 we show why the time-of-arrival operator for a free particle is not self-adjoint, and explore the possible modifications that can be made in order to make it self-adjoint. The idea is that by modifying the operator in a very small neighborhood around $k = 0$, one can formally construct a modified time-of-arrival operator which behaves in much the same way as the unmodified time-of-arrival operator.

We then explore some of the properties of the modified time-of-arrival states. In Section 4.5 we examine normalizable states which are coherent superpositions of time-of-arrival eigenstates, and discuss the possibility of localizing these states at the location of arrival at the time-of-arrival. Our results for the "unmodified" part of the time-of-arrival state seem to agree with those of Muga, Leavens and Palao who have studied these states independently [30]. In Section 4.6 we show that in an eigenstate of the modified time-of-arrival operator, the particle, at the predicted time-of-arrival, is found far away from the point of arrival with probability 1/2. We also calculate the average energy of the states, in order to relate them to our proposal in Chapter 3 that one cannot measure the time-of-arrival to an accuracy better than $1/\bar{E}_k$. We end with concluding remarks in Section 4.7.

4.2 Conditions on A Time-of-Arrival Operator

As discussed in the previous section, although a direct measurement of the time-of-arrival may not be possible, one can still try to observe it indirectly by measuring some operator $\mathbf{T}(\mathbf{p}, \mathbf{x}, x_A)$. In the next two sections we shall examine this operator and its relation to the continuous measurements described in the previous chapters. First in this section we show that an exact time-of-arrival operator cannot exist for systems with bounded Hamiltonian.

To begin with, let us start with the assumption that the time-of-arrival is described, as other observables in quantum mechanics, by a Hermitian operator $\mathbf{T}$.

$$\mathbf{T}(t)|t_A\rangle_t = t_A|t_A\rangle_t \tag{4.95}$$

Here the subscript $\rangle_t$ denotes the time dependence of the eigenkets, and $\mathbf{T}$ may depend explicitly on time. Hence for example, the probability distribution for the time-of-arrival for the state

$$|\psi\rangle = \int g(t'_A)|t'_A\rangle dt'_A \tag{4.96}$$

will be given by $prob(t_A) = |g(t_A)|^2$. We shall now also assume that the spectrum of $\mathbf{T}$ is continuous and unbounded: $-\infty < t_A < \infty$.

Should $\mathbf{T}$ correspond to time-of-arrival it must satisfy the following obvious condition. $\mathbf{T}$ must be a constant of motion and in the Heisenberg representation

$$\frac{d\mathbf{T}}{dt} = \frac{\partial \mathbf{T}}{\partial t} + \frac{1}{i}[\mathbf{T}, H] = 0. \tag{4.97}$$

That is, the time-of-arrival cannot change in time. If, for example, I measure that the bus is supposed to arrive at 7 p.m., then if I make another measurement at some other time, I should still find that the bus should (or did) arrive at 7 p.m

For a time-independent Hamiltonian, time translation invariance implies that the eigenkets $|t_A\rangle_t$ depends only on $t - t_A$, i.e. the eigenkets cannot depend on the absolute time t. This means for example that at the time of arrival: $|t_A\rangle_{t=t_A} = |t'_A\rangle_{t=t'_A}$. Time-translation invariance implies

$$|t_A\rangle_t = e^{-i\mathbf{G}}|0\rangle_0. \tag{4.98}$$

where $\mathbf{G} = \mathbf{G}(t - t_A)$ is a hermitian operator. Therefore, $|t_A\rangle_t$ satisfies the differential equations

$$i\frac{\partial}{\partial t_A}|t_A\rangle_t = \frac{\partial \mathbf{G}}{\partial t_A}|t_A\rangle_t = -\frac{\partial \mathbf{G}}{\partial t}|t_A\rangle_t, \qquad i\frac{\partial}{\partial t}|t_A\rangle_t = \frac{\partial \mathbf{G}}{\partial t}|t_A\rangle_t. \tag{4.99}$$

Now act on the eigenstate equation (4.95) with the differential operators $i\partial_{t_A}$ and $i\partial_t$. This yields

$$-\mathbf{T}\frac{\partial \mathbf{G}}{\partial t}|t_A\rangle_t = -t_A\frac{\partial \mathbf{G}}{\partial t}|t_A\rangle_t + i|t_A\rangle_t, \tag{4.100}$$

and

$$i\frac{\partial \mathbf{T}}{\partial t}|t_A\rangle_t + \mathbf{T}\frac{\partial \mathbf{G}}{\partial t}|t_A\rangle_t = t_A\frac{\partial \mathbf{G}}{\partial t}|t_A\rangle_t. \tag{4.101}$$

By adding the two equations above, the dependence on $\partial\mathbf{G}/\partial t$ drops off, and after using the constancy of $\mathbf{T}$ (eq. 4.97) we get

$$\Big([\mathbf{T}, H] + i\Big)|t_A\rangle = 0. \tag{4.102}$$

Since the eigenkets $|t_A\rangle$ span, by assumption, the full Hilbert space

$$[\mathbf{T}, H] = -i. \tag{4.103}$$

Hence $\mathbf{T}$ is a generator of energy translations. From equation (4.97) we have $\mathbf{T} = t - \hat{\mathbf{T}}$, where $\hat{\mathbf{T}}$ is the "time operator" of the system whose Hamiltonian is H. It is well know that equation (4.103) is inconsistent unless the Hamiltonian is unbounded from above and below [8].

4.3 Time-of-Arrival Operators vs. Continuous Measurements

Although formally there cannot exist a time-of-arrival operator $\mathbf{T}$, it may be possible to approximate $\mathbf{T}$ to arbitrary accuracy [9]. This modified operator will be discussed more fully, in the next section, but for now, assume that we can define the regularized Hermitian operator

$$\mathbf{T}' = O(\mathbf{p})T\mathbf{O}(\mathbf{p}) \tag{4.104}$$

where $O(\mathbf{p})$ is a function which is equal to 1 at all values of p except around a small neighborhood of $k = 0$. For $|p| < \epsilon$, $O(\mathbf{p})$ goes rapidly to zero (at least as fast as $\sqrt{k}$). $\mathbf{T}'$ is thus an operator which behaves just like $\mathbf{T}$ except in a very small neighborhood around $k = 0$.

It's eigenvalues are complete and orthogonal, and it circumvents the proof given above, because it satisfies

$$[\mathbf{T}', \mathbf{H}] = -i\mathbf{O} \tag{4.105}$$

i.e. it is not conjugate to H at $p = 0$. Although $\mathbf{T}$ is not always the shift operator of the energy, the measurement can be carried out in such a way that this will not be of consequence. To see this, consider the interaction Hamiltonian

$$H_{meas} = \delta(t)\mathbf{q}\mathbf{T}', \tag{4.106}$$

which modifies the initial wave function $\psi \to \exp(-iqT')\psi$. We need to demand that $\mathbf{T}'$ acts as a shifts operator of the energy of ψ during the measurement. Therefore we need that $q > -E_{min}$, where E_{min} is the minimal energy in the energy distribution of ψ. In this way, the measurement does not shift the energy down to $E = 0$ where $\mathbf{T}'$ is no longer conjugate to H. The value of $\mathbf{T}'$ is recorded on the conjugate of q – call it P_q. Now the uncertainty is given by $dT'_A = d(P_q) = 1/dq$, thus naively from $dq = 1/dT'_A < E_{min}$, we get $E_{min}dT' > 1$. However here, the average $\langle q \rangle$ was taken to be zero. There is no reason not to take $\langle q \rangle$ to be much larger than E_{min}, so that $\langle q \rangle - dq >> -E_{min}$. If we do so, the measurement increases the energy of ψ and $\mathbf{T}'$ is always conjugate to H. The limitation on the accuracy is in this case $dT'_A > 1/\langle q \rangle$ which can be made as small as we like.

However, even small deviations from the commutation relation (4.103) are problematic. Not only is the modification arbitrary, it will also result in inaccurate measurements. For example, since

$$\frac{d\mathbf{T}'}{dt} = \mathbf{1} - \mathbf{O}, \tag{4.107}$$

$$\mathbf{T}'(t) = \mathbf{T}'(0) - t(\mathbf{1} - \mathbf{O}). \tag{4.108}$$

For the component of the wave function $\tilde{\psi}(k)$ which has support in the neighborhood of $k = 0$, the time-of-arrival will no longer be a constant of motion. The average value of $\mathbf{T}'(t)$ for the state $\tilde{\psi}(k)$ is given by

$$\langle \mathbf{T}(t) \rangle = \langle \mathbf{T}(0) \rangle - t \int dk\ [1 - O(k)]\, |\tilde{\psi}(k)|^2 \,. \tag{4.109}$$

The second term on the right hand side will be non-zero if $\tilde{\psi(k)}$ has support for $|k| < \epsilon$. Even if $\tilde{\psi}(k)$ is negligibly small around $k = 0$, the second term will grow with time. Thus, one only needs to wait a sufficiently long period of time before measuring $\mathbf{T}'$ to find that the average time-of-arrival will change in time. As mentioned in the previous section, this does not correspond to what one would want to call a “time-of-arrival”. The greater $|\tilde{\psi}(k)|^2$ is around $k = 0$, the greater the deviation from the condition that the time-of-arrival be a constant of the motion. Another difficulty with the time-of-arrival operator, is that if one makes a measurement at a time t' before the particle arrives, then one needs to know the full Hamiltonian from time t' until t_A. Even if one knows the full Hamiltonian, and can find an approximate time-of-arrival operator, one has to have faith that the Hamiltonian will not be perturbed after the measurement has been made. On the other hand, the continuous measurements we have described can be used with any Hamiltonian.

A further difficulty is that a measurement of the time-of-arrival operator is not equivalent to continuously monitoring the point-of-arrival I.e., measuring the time-arrival-operator is not equivalent to the measurement procedures discussed in Chapter 2. If $\mathbf{P}_0$ is the projector onto $x = 0$, one finds that

$$\langle\psi[\mathbf{T}, \mathbf{P}_0]|\psi\rangle = -\frac{i}{2} Re\{\psi(x=0) \int dk \psi^*(k) \frac{m}{k^2}\}. \tag{4.110}$$

A measurement of the time-of-arrival operator does not commute with the projection operator onto the point of arrival.

It is also clear from the discussion in Section 3.3, that in the limit of high precision, continuous measurements respond very differently to the time operator. When measurements are made with physical clocks, then in the limit $dt_A \to 0$ all the particles bounce back from the detector. Such a behavior does not occur for the time of arrival operator which can be measured to arbitrary accuracy. Nevertheless, one may still hope that since the eigenstates of $\mathbf{T}$ have an infinitely spread in energy, they do trigger a clock even if $dt_A \to 0$. For the type of models we have been considering, we can show however that this will not happen.

Let us assume that the interaction of one eigenstate of $\mathbf{T}$ with the clock (of, say, Section 3.3.1) evolves as

$$|t_A\rangle|y = t_0\rangle \ \rightarrow \ |\chi(t_A)\rangle|y = t_A\rangle + |\chi'(t_A)\rangle|y = t\rangle. \tag{4.111}$$

Here, $|y = t_0\rangle$ denotes an initial state of the clock with $dt_A \to 0$, $|\chi(t_A)\rangle$ denotes the final state of the particle if the clock has stopped, and $|\chi'(t_A)\rangle$ the final state of the particle if the clock has not stopped.

Since the eigenstates of $\mathbf{T}$ form a complete set, we can express any state of the particle as $|\psi\rangle = \int dt_A C(t_A)|t_A\rangle$. We then obtain :

$$\int dt_A C(t_A)|t_A\rangle|y=t_0\rangle \;\rightarrow\; \int dt_A C(t_A)|\chi(t_A)\rangle|y=t_A\rangle + \left(\int dt_A C(t_A)|\chi'(t_A)\rangle\right)|y=t\rangle. \tag{4.112}$$

The final probability to measure the time-of-arrival is hence $\int dt_a|C(t_a)\chi(t_a)|^2$. On the other hand we found that for a general wave function ψ, in the limit of $dt_a \rightarrow 0$, the probability for detection vanishes. Since the states of the clock, $|y=t_a\rangle$, are orthogonal in this limit, this implies that $\chi(t_a)=0$ in eq. (4.111) for all t_A. Therefore, the eigenstates of $\mathbf{T}$ cannot trigger the clock.

It should be mentioned however, that one way of circumventing this difficulty may be to consider a coherent set of $\mathbf{T}$ eigenstates states instead of the eigenstates themselves. These normalizable states will no longer be orthogonal to each, so they may trigger the clock if they have sufficient energy [1]. In this regard it is of interest to prematurely quote a result which we will show in Section 4.5 - the average energy of a Gaussian distribution of time-of- arrival eigenstates is proportional to $1/\Delta$ where Δ is the spread of the Gaussian. This puts us at the edge of the limitation given in Equation (3.91).

[1] An arbitrary wave packet can be written as a superposition of normalized eigenstates, and yet we know that arbitrary wave packets do not trigger the clocks of Section 3.3.1. This creates a somewhat interesting situation if normalized eigenstates trigger a clock, but wave packets made of superpositions of them do not.

4.4 The Modified Time-of-Arrival Operator

Kinematically, one expects that the time-of-arrival operator for a free particle arriving at the location $x_A = 0$ might be given by

$$\mathbf{T} = -\frac{m}{2}\frac{1}{\sqrt{\mathbf{p}}}\mathbf{x}(0)\frac{1}{\sqrt{\mathbf{p}}}. \tag{4.113}$$

The operator $-m(\frac{1}{\mathbf{p}}\mathbf{x} + \mathbf{x}\frac{1}{\mathbf{p}})$ is equivelant to the one above as can be seen by use of the commutation relations for $\mathbf{x}$ and $\mathbf{p}$.

In the k representation this operator can be written as

$$\mathbf{T}(k) = -im\frac{1}{\sqrt{k}}\frac{d}{dk}\frac{1}{\sqrt{k}} \quad = -im(\frac{1}{k}\frac{d}{dk} + \frac{d}{dk}\frac{1}{k}) \tag{4.114}$$

where $\sqrt{k} = i\sqrt{|k|}$ for $k < 0$. If one solves the eigenvalue equation, one finds a set of anti-symmetric states [2] for this operator given by

$$g^a_{t_A}(k) = (\theta(k) - \theta(-k))\,\frac{1}{\sqrt{2\pi m}}\sqrt{|k|}e^{i\frac{t_A k^2}{2m}} \quad . \tag{4.115}$$

The symmetric states are

$$g^s_{t_A}(k) = (\theta(k) + \theta(-k))\,\frac{1}{\sqrt{2\pi m}}\sqrt{|k|}e^{i\frac{t_A k^2}{2m}} \tag{4.116}$$

However, the operator is not self-adjoint and these states are not orthogonal.

$$\langle t'_A|t_A\rangle = \frac{1}{2\pi m}\int_0^\infty dk^2\, e^{\frac{i}{2m}k^2(t_A - t'_A)} \quad = \delta(t_A - t'_A) - \frac{i}{\pi(t_A - t'_A)}. \tag{4.117}$$

It is important to recall that a symmetric operator which is not self-adjoint always has complex eigenvalues and eigenfunctions [45]. If in (4.115) we choose t_A complex, having positive imaginary part, then the eigenstate is a square integrable function (i.e.. it is a true eigenstate of the operator) which has complex eigenvalues.

[2] To find the actual generalized eigenfunctions of this operator, one needs to first specify a domain of definition. We will not deal with this issue here, since the modified time-of-arrival operator is self-adjoint, and therefore one does not encounter the same problems as with the time-of-arrival operator which is not self-adjoint.

Trying to make $\mathbf{T}$ self-adjoint by defining boundary conditions at $k = 0$ leads to the requirement on square integrable wave functions $u(k)$,$v(k)$ such that

$$\langle u, \mathbf{T}v\rangle - \langle \mathbf{T}^* u, v\rangle = i\,m \left[\lim_{k\to 0^-} \frac{v(k)\overline{u(k)}}{|k|} + \lim_{k\to 0^+} \frac{v(k)\overline{u(k)}}{|k|} \right] = 0 \tag{4.118}$$

i.e.. the boundary conditions must be chosen so that $\frac{v(k)\overline{u(k)}}{k}$ is continuous through $k = 0$. This continuity condition cannot force $u(k)$ to have the same boundary conditions as $v(k)$ for any choice of boundary condition on $v(k)$. For example, if we choose $v(k)/\sqrt{k}$ to be continuous through the origin, then $u(k)/\sqrt{k}$ must be anti-continuous through the origin. I.e. the domain of definition of $\mathbf{T}$ and $\mathbf{T}^*$ differ and $\mathbf{T}$ cannot be self-adjoint. This is not at all surprising, given the proof in Section 4.2.

One might however, try to modify $\mathbf{T}$ in order to make it self-adjoint in the manner shown in Section 4.3 [9]. Consider the operator

$$\mathbf{T}_\epsilon(k) = -im\sqrt{f_\epsilon(k)}\frac{d}{dk}\sqrt{f_\epsilon(k)} \tag{4.119}$$

where $f_\epsilon(k)$ is some smooth function which differs from $1/k$ only near $k = 0$. Since $u(k)$ and $v(k)$ could diverge at the origin at a rate approaching $1/\sqrt{k}$ and still remain square-integrable, if $f_\epsilon(k)$ goes to zero at least as fast as k, then $\mathbf{T}_\epsilon$ will be self-adjoint and defined over all square integrable functions. However, as we show in Sections 4.5 and 4.6, these eigenstates do not behave as one would expect a time of arrival eigenstate to behave.

It can be verified that $\mathbf{T}_\epsilon$ has a degenerate set of eigenstates $|t_A, +\rangle$ for $k > 0$ and $|t_A, -\rangle$ for $k < 0$, given by

$$g^{\pm}_{t_A}(k) = \langle k|t_A, \pm\rangle = \theta(\pm k)\frac{1}{\sqrt{2\pi m}}\frac{1}{\sqrt{f_\epsilon(k)}}e^{\frac{it_A}{m}\int_{\pm\epsilon}^{k}\frac{1}{f_\epsilon(k')}dk'} \tag{4.120}$$

which are orthonormal as expected. Grot, Rovelli, and Tate [9] choose to work with the states given by

$$f_\epsilon(k) = \begin{cases} \frac{k}{\epsilon^2} & |k| < \epsilon \\ \frac{1}{k} & |k| > \epsilon \end{cases} \tag{4.121}$$

If $\epsilon \to 0$, one might expect $\mathbf{T}_\epsilon$ to be a good approximation to the time of arrival operator when acting on states that do not have support around $k = 0$ [9].

As we show in Appendix C, when these states are examined in the x-representation, and if one only considers the contribution to the Fourier transform of $g^+_{t_A}(k)$ from $|k| > \epsilon$ (i.e.. the "unmodified" part of the eigenstate), then one finds that at the time-of-arrival, the states are not delta functions $\delta(x)$ but are proportional to $x^{-3/2}$; they have support over all x. However, although the state has long tails out to infinity, the quantity $\int dx' |x'^{-3/2}|^2 \sim x^{-2}$ goes to zero as $x \to \infty$. Furthermore, the modulus squared of the eigenstates diverges when integrated around the point of arrival $x = 0$. As a result, one might expect that the normalized state will be localized at the point-of-arrival at the time-of-arrival. In Section 4.5 we show that this is indeed so. However, the full eigenstate, is made up both of this "unmodified" piece, and a modified piece. The modified part of the eigenstate is not well localized at the time-of-arrival. The contribution to the Fourier-transform of the state $g^+_{t_A}(k)$ from $0 < k < \epsilon$ is given by

$${}_\epsilon\tilde{g}^+(x)_{t_A} = \frac{\epsilon}{\sqrt{2\pi m}} \int_0^\epsilon \frac{dk}{\sqrt{k}} e^{ikx} e^{-it_A \frac{k^2}{2m}} e^{\frac{i\epsilon^2 t_A}{m} \ln \frac{k}{\epsilon}} \, . \tag{4.122}$$

Because $\mathbf{T}_\epsilon$ is no longer the generator of energy translations for $|k| < \epsilon$, $g^+_{t_A}(k)$ is not time-translation invariant. For the $t_A = 0$ state, (4.4) can be integrated to give

$${}_\epsilon\tilde{g}^+(x)_{t_A} = \frac{\epsilon}{\sqrt{2xim}} \Phi(\sqrt{i\epsilon x}) \tag{4.123}$$

where Φ is the probability integral. For large x, ${}_\epsilon\tilde{g}^+(x)_{t_A}$ goes as $\frac{1}{\sqrt{x}}$ and the quantity $\int dx' |{}_\epsilon\tilde{g}^+_{t_A}(x')|^2 \sim \ln x$ diverges as $x \to \infty$. For small x, ${}_\epsilon\tilde{g}^+_{t_A}(x)$ is proportional to $e^{-i\epsilon x}$.

Its modulus squared vanishes when integrated around a small neighborhood of $x = 0$. ${}_\epsilon\tilde{g}^+(x)_{t_A}$ then, is not localized around the point of arrival, at the time-of-arrival. This will also be verified in the next section where we examine normalizable states. Although ${}_\epsilon\tilde{g}^+(x)_{t_A}$ is not localized around the point of arrival at the time of arrival, one might hope that this part of the state does not contribute significantly in time-of-arrival measurements when $\epsilon \to 0$. However, we will now see that for coherent superpositions of these eigenstates, half the norm is made up of the modified piece of the eigenstate.

4.5 Normalized Time-of-Arrival States

Since the time-of-arrival states are not normalizable, we will examine the properties of states $|\tau_\Delta\rangle$ which are narrow superpositions of the modified time-of-arrival eigenstates. These states are normalizable, although they are no longer orthogonal to each other [3] .

We can now consider coherent superpositions of these eigenstates

$$|\tau_\Delta^\pm\rangle = N\int dt_A |t_A, \pm\rangle e^{-\frac{(t_A-\tau)^2}{\Delta^2}}. \tag{4.124}$$

where N is a normalization constant and is given by $N = (\frac{2}{\pi\Delta^2})^{1/4}$. The spread dt_A in arrival times is of order Δ.

We now examine what the state $\tau(x,t)^+ = \langle x|\tau_\Delta^+\rangle$ looks like at the point of arrival as a function of time. In what follows, we will work with the state centered around $\tau = 0$ for simplicity. This will not affect any of our conclusions. $\tau^+(x,t)$ is given by

$$\begin{aligned}
\tau^+(x,t) &= N\int \langle x|e^{\frac{-i\mathbf{p}^2 t}{2m}}|t_A,+\rangle e^{-\frac{t_A^2}{\Delta^2}} dt_A \\
&= N\int_0^\epsilon e^{-\frac{t_A^2}{\Delta^2}} e^{\frac{-ik^2}{2m}t} e^{ikx} g_{t_A}^+(k) dt_A\, dk \quad + N\int_\epsilon^\infty e^{-\frac{t_A^2}{\Delta^2}} e^{\frac{-ik^2}{2m}t} e^{ikx} g_{t_A}^+(k) dt_A\, dk \\
&\equiv {}_\epsilon\tau^+(x,t) \ + {}_o\tau^+(x,t)
\end{aligned} \tag{4.125}$$

[3] These coherent states form a positive operator valued measure (POVM). The measurement of time-of-arrival using POVMs has been discussed in [26].

As argued in the previous section, the second term should act like a time-of-arrival state. The first term is due to the modification of **T** and has nothing to do with the time of arrival. We will first show that the second term can indeed be localized at the point-of-arrival $x = 0$ at the time of arrival $t = t_A$. We will do this by expanding it around $x = 0$ in a Taylor series. After taking the limit $\epsilon \to 0$, it's n'th derivative at $x = 0$ is given by

$$\begin{aligned} \frac{d^n}{dx^n}{}_o\tau^+(x,t)|_{x=0} &= \frac{N}{\sqrt{2\pi m}}\int\int_\epsilon^\infty e^{-\frac{t_A^2}{\Delta^2}}\theta(k)\sqrt{k}(ik)^n e^{\frac{ik^2}{2m}(t_A-t)}dt_A\,dk \\ &= \frac{N\Delta}{\sqrt{2m}}i^n\int_0^\infty e^{\frac{-k^4\Delta^2}{16m^2}}e^{\frac{-ik^2t}{2m}}k^{\frac{1}{2}+n}dk \\ &= \frac{2^{\frac{3}{8}+\frac{3n}{4}}i^n}{\pi^{\frac{1}{4}}}\Gamma(\frac{3}{4}+\frac{n}{2})(\frac{m}{\Delta})^{\frac{1}{4}+\frac{n}{2}}e^{-\frac{t^2}{2\Delta^2}}D_{-\frac{3}{4}-\frac{n}{2}}(\frac{it\sqrt{2}}{\Delta}) \end{aligned} \tag{4.126}$$

where $D_p(z)$ are the parabolic-cylinder functions. For any finite t, we can choose Δ small enough so that the argument of $D_p(z)$ is large, and can be written as

$$D_p(z) \simeq e^{-\frac{z^2}{4}}z^p(1 - \frac{p(p-1)}{2z^2} + \cdots) \quad . \tag{4.127}$$

We can now write ${}_o\tau^+(0,t)$ as a Taylor expansion around $x = 0$

$${}_o\tau^+(x,t) \simeq \sqrt{\Delta}(\frac{m}{t^3})^{\frac{1}{4}}\sum_{n=0}^{\infty}a_n(\sqrt{\frac{m}{t}}x)^n \tag{4.128}$$

where a_n is a numerical constant given by

$$a_n = i^{-\frac{3}{4}+\frac{n}{2}}2^{\frac{n-1}{2}}\pi^{-\frac{1}{4}}\Gamma(\frac{3}{4}+\frac{n}{2}) \tag{4.129}$$

We can now see that for any finite t the amplitude for finding the particle around $x = 0$ goes to zero as Δ goes to zero. The probability of being found at the point of arrival at a time other than the time-of-arrival can be made arbitrarily small. On the other hand, at the time-of-arrival $t = 0$, we will now show that the state ${}_o\tau^+(x,t)$ can be as localized as one wishes around $x = 0$.

From (4.126), we expand ${}_o\tau^+(x,0)$ as a Taylor series

$${}_o\tau^+(x,0) = (\frac{m}{\Delta})^{\frac{1}{4}}\sum_{n=0}^{\infty}b_n(\sqrt{\frac{m}{\Delta}}x)^n \tag{4.130}$$

where

$$b_n = i^n 2^{n-\frac{3}{4}} \pi^{-\frac{1}{4}} \Gamma(\frac{3}{8} + \frac{n}{4}) \tag{4.131}$$

We see then that ${}_o\tau^+(x,0)$ is a function of $\sqrt{\frac{m}{\Delta}}x$ (with a constant of $(\frac{m}{\Delta})^{1/4}$ out front). As a result, the probability of finding the particle in a neighborhood δ of x is given by

$$\int_{-\delta}^{\delta} |{}_o\tau^+(\sqrt{\frac{m}{\Delta}}x,0)|^2 dx = \sqrt{\frac{\Delta}{m}} \int_{-\delta\sqrt{\frac{m}{\Delta}}}^{\delta\sqrt{\frac{m}{\Delta}}} |{}_o\tau^+(u,0)|^2 du. \tag{4.132}$$

Since $|{}_o\tau^+(u,0)|^2$ is proportional to $\sqrt{\frac{m}{\Delta}}$, and is square integrable, we see that for any δ, one need only make Δ small enough, in order to localize the entire particle in the region of integration. ${}_o\tau^+(x,t)$ is localized in a neighborhood δ around the point-of-arrival at the time-of-arrival as $\Delta \to 0$. The state is localized in a region δ of order $\sqrt{\frac{\Delta}{m}}$. This is what one would expect from physical grounds, since we have

$$\begin{aligned} dx &\sim dt_A \frac{\langle k \rangle}{m} \\ &\sim \sqrt{\frac{\Delta}{m}}\,. \end{aligned} \tag{4.133}$$

($\langle k \rangle$ is calculated in the following section and is proportional to $\sqrt{m/\Delta}$). The probability distribution of ${}_o\tau^+(x,t)$ at $t=\tau$ is shown in Figure 4.1. This behavior of the unmodified piece of the time-of-arrival state, ${}_o\tau^+(x,t)$, as a function of time appears to agree with the results of Muga, Leavens and Palao, who have studied these coherent states independently [30].

The modified part of the time-of-arrival state, ${}_\epsilon\tau^+(x,0)$, is not found near the origin at $t = t_A = 0$. We find

$$\begin{aligned} {}_\epsilon\tau^+(x,0) &= N\frac{\epsilon}{\sqrt{2\pi m}} \int_{-\infty}^{\infty} \int_0^{\epsilon} e^{-\frac{t_A^2}{\Delta^2}} \frac{1}{\sqrt{k}} e^{\frac{i\epsilon^2 t_A}{m} \ln\frac{k}{\epsilon}} e^{ikx} dk\, dt_A \\ &= N\frac{\epsilon^{3/2}}{\sqrt{2\pi m}} \int_{-\infty}^{\infty} e^{-\frac{t_A^2}{\Delta^2}} \gamma(\frac{i\epsilon^2 t_A}{m} + \frac{1}{2}, -i\epsilon x)(-i\epsilon x)^{-\frac{1}{2}-\frac{i\epsilon^2 t_a}{m}} dt_A. \end{aligned} \tag{4.134}$$

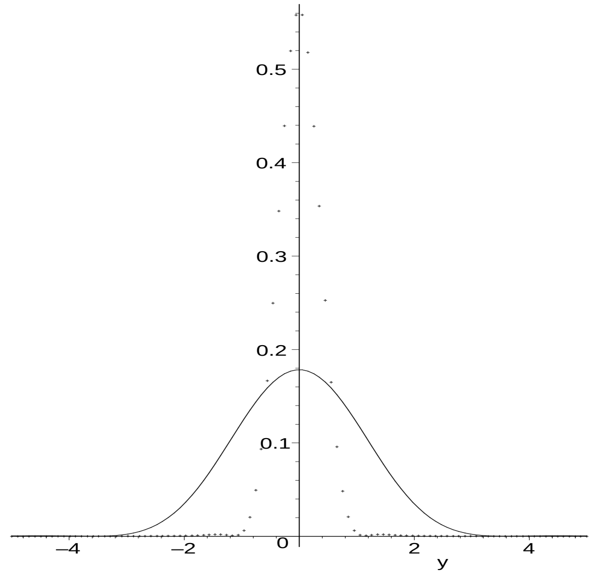

Figure 4.1: Unmodified part of time-of-arrival eigenstate. $|_o\tau^+(x,\tau)|^2$ vs. x, with $\Delta = m$ (solid line), and $\Delta = \frac{m}{10}$ (dashed line). As Δ gets smaller, the probability function gets more and more peaked around the origin.

If $i\epsilon x$ is not large, we can use the fact that for Δ and ϵ very small, $i\epsilon^2 t_A/m \ll 1/2$ so that we have

$$ {}_\epsilon\tau^+(x,0) \simeq (2\pi)^{\frac{1}{4}}\sqrt{\frac{\epsilon^3\Delta}{2m}}\frac{\Phi(\sqrt{-i\epsilon x})}{\sqrt{-i\epsilon x}}\,. \tag{4.135} $$

Note the similarity between this state (the form above is not valid for large x), and that of the modified part of the eigenstate (4.123). We are interested in the case where $\frac{\epsilon^2\Delta}{m}$ goes to zero, in which case ${}_\epsilon\tau^+(x,0)$ vanishes near the origin. For large ϵx, it goes as $\sqrt{\frac{\epsilon^2\Delta}{xm}}$. From (4.134) we can also see that if $\epsilon x > e^{\frac{m}{\epsilon^2\Delta}}$ then the last factor in the integrand oscillates rapidly and the integral falls rapidly for larger x. Thus, as we make $\frac{\epsilon^2\Delta}{m}$ smaller, the value of the modulus squared decrease around $x = 0$, but the tails, which extend out to $e^{\frac{m}{\epsilon^2\Delta}}/\epsilon$, get longer. $\int^x |_\epsilon\tau^+(x,0)|^2$ goes as $\frac{\epsilon^2\Delta}{m}\ln x$ up to $\epsilon x \sim e^{\frac{m}{\epsilon^2\Delta}}$.

As $\frac{\epsilon^2\Delta}{m} \to 0$, the particle is always found in the far-away tail. The state ${}_\epsilon\tau^+(x,0)$ is not found near the point of arrival at the time-of-arrival. It's probability distribution at

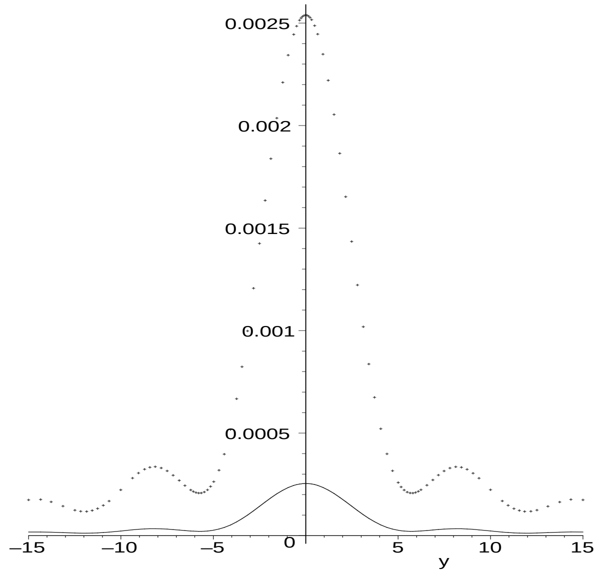

Figure 4.2: Modified part of time-of-arrival eigenstate. $\frac{1}{\epsilon}|_\epsilon\tau^+(x,\tau)|^2$ vs. ϵx, of $\sqrt{\frac{m}{\Delta}}x$. with $\Delta\epsilon^2 = \frac{m}{10}$ (solid line) and $\Delta\epsilon^2 = \frac{m}{100}$ (dashed line). As Δ or ϵ gets smaller, the probability function drops near the origin, and grows longer tails which are exponentially far away.

$t = t_A = 0$ is shown in Figure 4.2.

4.6 Contribution to the Norm due to Modification of T

We now show that the modified part of $|\tau_\Delta^+\rangle$ contains at least half the norm, no matter how small ϵ is made. The norm of the state $|\tau_\Delta^+\rangle$ can be written as

$$\begin{aligned} \int |\langle k|\tau_\Delta^+\rangle|^2 dk &= N^2\int_0^\epsilon |e^{-\frac{t_A^2}{\Delta^2}} g_{t_A}^+(k)dt_A|^2 dk \ + N^2\int_\epsilon^\infty |e^{-\frac{t_A^2}{\Delta^2}} g_{t_A}^+(k)dt_A|^2 dk \\ &\equiv N_\epsilon^2 + N_o^2 \end{aligned} \tag{4.136}$$

where N_ϵ^2 is the norm of the modified part of the time-of-arrival state, and N_o^2 is the norm of the unmodified part. The second term can be integrated to give

$$\begin{aligned} N_o^2 &= \frac{N^2}{2\pi m}\int\int_\epsilon^\infty e^{-\frac{t_A^2+t_A'^2}{\Delta^2}} e^{i\frac{k^2-\epsilon^2}{2m}(t_A-t_A')} dt_A dt_A' dk \\ &= \frac{N^2\Delta^2\pi}{m}\int_0^\infty d\tilde{k}\,\tilde{k}e^{\frac{-\tilde{k}^4\Delta^2}{8m^2}} \end{aligned}$$

$$= \frac{1}{2} \tag{4.137}$$

where without loss of generality, we are looking at the state centered around $\tau = 0$ at $t = 0$.

The unmodified piece can contain only half the norm. The rest is found in the modified piece.

$$\begin{aligned} N_\epsilon^2 &= \frac{N^2}{2\pi m}\int_0^\epsilon dk \int dt_A dt'_A \frac{\epsilon^2}{k} e^{\frac{-t_A^2 - t_A'^2}{\Delta^2}} e^{i\epsilon^2 \ln\frac{k}{\epsilon}\frac{t'_A - t_A}{m}} \\ &= \frac{N^2\Delta^2}{2m}\int_0^\epsilon dk e^{\frac{-\epsilon^4\Delta^2 \ln^2 k/\epsilon}{2m^2}} \frac{\epsilon^2}{k} \\ &= \frac{1}{2} \end{aligned} \tag{4.138}$$

The reason for this, is that essentially, the modification $1/k \to f_\epsilon(k)$ involves expanding the region $0 < k < \epsilon$ into the entire negative k-axis. I.e. we see from (4.117) that in order to make the eigenstates orthogonal, one needs the integration variable to go from $-\infty$ to ∞ and this involves making the modification

$$k^2 \to z^\pm = \int_{\pm\epsilon}^k \frac{dk'}{f_\epsilon(k')} \ . \tag{4.139}$$

The orthogonality condition then becomes

$$\langle t'_A, \pm | t_A, \pm\rangle = \int_{-\infty}^\infty dz^\pm \frac{1}{2\pi m} e^{i(t_A - t'_A)\frac{z^\pm}{m}} = \delta(t_A - t'_A) \ . \tag{4.140}$$

No matter how small we make ϵ, half the norm comes from the contribution $z^\pm < 0$ which is the modified part of the eigenstate. As a result, if one makes a measurement of time-of-arrival, then one finds that half the time, the particle is not found at the point of arrival at the predicted time-of-arrival. Modified time of arrival states do not always arrive on time.

From (4.138), one can also see that if $f_\epsilon(k)$ goes to zero faster than k, then N_ϵ will diverge as Δ or ϵ go to zero. If $f_\epsilon(k) = k^{1+\delta}$, then we find

$$N_\epsilon = \frac{1}{2} e^{\frac{\delta^2 m^2}{2\epsilon^4\Delta^2}} \left[1 - \Phi(\frac{-\delta\epsilon^2\Delta\sqrt{2}}{m})\right] \tag{4.141}$$

As ϵ or Δ go to zero, N_ϵ diverges, and if we renormalize the state, the entire norm will be made up of the modified part of the eigenstate.

It is also of interest to calculate the average value of the kinetic energy for these states, in order to see whether these states will trigger the physical clocks discussed in Chapter 3. In calculating the average energy, the modified piece will not matter since k^2 goes to zero at $k = 0$ faster than $\frac{1}{\sqrt{k}}$ diverges. We find

$$\begin{aligned}\langle\tau_\Delta^+|\mathbf{H}_k|\tau_\Delta^+\rangle &= \int dk\frac{k^2}{2m}\langle\tau_\Delta^+|k\rangle\langle k|\tau_\Delta^+\rangle \\ &= \frac{N^2}{\pi(2m)^2}\int_0^\infty k^3 e^{\frac{i(t_A-t'_A)k^2}{2m}} e^{-\frac{t_A^2+t_A'^2}{\Delta^2}} dt_A\, dt'_A\, dk \\ &= \frac{4}{\Delta\sqrt{2\pi}} \end{aligned} \tag{4.142}$$

We see therefore, that the kinematic spread in arrival times of these states is proportional to $1/\bar{E}_k$. Since the probability of triggering the model clocks discussed in Chapter 3 decays as $\sqrt{E_k \delta t_A}$, where δt_A is the accuracy of the clock, we find that the states $|\tau_\Delta^+\rangle$ will not always trigger a clock whose accuracy is $\delta t_A = \Delta$.

4.7 Limited Physical Meaning of Time-of-Arrival Operators

We have seen that formally, a time-of-arrival operator cannot exist. If one modifies the time-of-arrival operator so as to make it self-adjoint, then its eigenstates no longer behave as one expects time-of-arrival states to behave. Half the time, a particle which is in a time-of-arrival state will not arrive at the predicted time-of-arrival. The modification also results in the fact that the states are no longer time-translation invariant.

For wave functions which don't have support at $k = 0$, measurements can be carried out in such a way that the modification will not effect the results of the measurement. Nonetheless, after the measurement, the particle will not arrive on time with a probability of 1/2. One cannot use $\mathbf{T}_\epsilon$ to prepare a system in a state which arrives at a certain time.

We would also like to stress that continuous measurements differ both conceptually and quantitatively from a measurement of the time-of-arrival operator. Operationally one performs here two completely different measurements. While the time-of-arrival operator is a formally constructed operator which can be measured by an impulsive von-Neumann interaction, it seems that continuous measurements are much more closer to actual experimental set-ups. Furthermore, we have seen that the result of these two measurements do not need to agree, in particular in the high accuracy limit, continuous measurements give rise to entirely different behavior. This suggests that as in the case of the problem of finding a “time operator” [20] for closed quantum systems, the time-of-arrival operator has a somewhat limited physical meaning.