3.1 A Limitation on Time-of-Arrival Measurements

In the previous Chapter, we saw that if we attempt to measure the time of an event using a rapid series of measurements, then our measurement will disturb the very thing which we are trying to measure. For the simple case of a two state system, we were able to show that if we make the measurement accurately enough, the system freezes, and the event never occurs.

In this Chapter, we study measurements of the time-of-arrival of a particle to a particular location using physical clocks. The clocks are coupled to the system in such a way that when the particle arrives to the fixed location, the clock will read the time-of-arrival. Unlike the continuous measurement procedure discussed in the last Chapter, we obtain the time of the event using only a single measurement which is made well after the event has occurred. Nonetheless, we will find that if we make the measurement extremely accurate, the measurement will fail.

Consider a free particle, upon which a measurement is performed to determine the time-of-arrival to $x = x_A$. The time-of-arrival can be recorded by a clock situated at $x = x_A$ which switches off when the particle reaches it. In classical mechanics we could, in principle, achieve this with the smallest non-vanishing interaction between the particle and the clock, and hence measure the time-of-arrival with arbitrary accuracy.

In classical mechanics there are other indirect methods to measure the time-of-arrival. One could invert the equation of motion of the particle and obtain the time in terms of the location and momentum, $T_A(x(t), p(t), x_A)$. This function can be determined at *any time* t, either by a simultaneous measurement of $x(t)$ and $p(t)$ and evaluation of T_A, or by a direct coupling to $T_A(x(t), p(t), x_A)$. One could also measure the time-of-arrival using the method discussed in the previous chapter. By using a weak interaction which doesn't disturb the system, one can continually monitor the point of arrival to see if the particle has arrived.

These different methods, namely, the direct measurement, indirect measurement, and continual monitoring are classically equivalent. They give rise to the same classical time-of-arrival. They are not equivalent however, in quantum mechanics

In quantum mechanics the corresponding operator $\mathbf{T_A}(\mathbf{x}(t), \mathbf{p}(t), x_A)$, if well defined, can in principle be measured to any accuracy. On the other hand, a direct measurement cannot determine the time-of-arrival of free particles to arbitrary accuracy (as was originally argued by Allcock[15]). In Section 3.3.2, we argue that Allcock's arguments are not sufficient to limit the accuracy of time-of-arrival measurements. One needs to consider models with physical clocks. Using these models, we shall argue that the accuracy of time-of-arrival measurements cannot be better than

$$\delta t_A > 1/E_k, \tag{3.39}$$

where E_k is the initial kinetic energy of the particle. The basic reason is that, unlike a classical clock, in quantum mechanics the uncertainty in the clock's energy grows when its accuracy improves [24]. We find that particles with initial kinetic energy E_k are reflected without switching off a clock if this clock is set to record the time-of-arrival with accuracy better than in eq. (3.39). Furthermore, for the small fraction of the ensemble that does manage to turn off the clock, the resulting probability distribution becomes distorted. A detailed discussion of direct time-of-arrival measurements will be discussed in Chapter 4.

We conclude in Section 3.3.5 with a discussion on why we expect eq. (3.39) to hold in general and we discuss the main results. An explicit calculation of the clock's final probability distribution is given in Appendix B.

3.2 Free Clocks

We will now attempt to make a measurement of the time-of-arrival. In order to do so, we will need a clock. An ideal clock is linear in time. I.e., the position of the clock's pointer should be proportional to the time t. It is not hard to see that an ideal clock can be represented by the Hamiltonian

$$\mathbf{H}_{clock} = \mathbf{P_y} \ . \tag{3.40}$$

To read the time of the clock, we measure the coordinate $\mathbf{y}$ conjugate to $\mathbf{P_y}$. Using the Heisenberg equations of motion we see that the variable $\mathbf{y}$ reads the correct parameter time t found in the Schrödinger equation.

$$\begin{aligned} \mathbf{y}(t) - \mathbf{y}(t_0) &= -i \int [\mathbf{y}, \mathbf{H}_{clock}] dt \\ &= t - t_0 \end{aligned} \tag{3.41}$$

The Hamiltonian for this clock is unbounded from above and below, nonetheless, using a sufficiently massive particle, we can approximate the ideal situation to arbitrary accuracy [1]. We write $\mathbf{p} = \langle \mathbf{p} \rangle + \delta \mathbf{p}$, and note that if $\mathbf{p} \gg \delta \mathbf{p}$, the free particle Hamiltonian is given by

$$\begin{aligned} \mathbf{H} &= \frac{\mathbf{p}^2}{2m} \\ &= \frac{\langle \mathbf{p} \rangle^2}{2m} + \frac{\langle \mathbf{p} \rangle \delta \mathbf{p}}{m} + O(\delta \mathbf{p}^2/m) \quad . \end{aligned} \tag{3.42}$$

Since the first term is constant and we can ignore the higher order term, we see that the Hamiltonian is approximately linear in momentum. The constant of proportionality is $\langle \mathbf{p} \rangle / m$ which we set to one for convenience. The corrections to equation (3.41) are also of order $\delta \mathbf{p}/\mathbf{p}$.

[1] One could also consider a Larmor clock with a bounded Hamiltonian $H_{clock} = \omega \mathbf{J}$ [16]. The Hilbert space is spanned by 2j+1 vectors where j is a natural number, and the clock's resolution can be made arbitrarily fine by increasing j.

From (3.41) we see that in order to use this clock to read the time, we need to know the initial position of the clock's dial $\mathbf{y}(t_0)$ and then subtract this from our final reading of $\mathbf{y}$. Quantum mechanics puts no limitation on how accurately this clock can be measured. If we want to accurately infer the time from the final reading of the clock then the clock must initially be prepared in a state with a very small uncertainty in y. At some later point, we can measures the coordinate $y(t_f)$ to any degree of accuracy we wish to infer the time from $y(t_f) - y(t_0)$. If initially dy was very small, then we know that the time is given by the final reading of y. However, if initially the state of the clock had a large spread in y, then the time we finally obtain will be inaccurate by an amount dy. This means that for this clock, the inaccuracy in the time measurement is given by

$$\delta T = dy \tag{3.43}$$

If we simply want to use this clock to read the time, then there are no restrictions on how accurate the clock can be. So far, nothing prevents us from making the initial state of the clock's pointer as close to an eigenstate of $y(t_0)$ as we desire. However, since $\mathbf{y}(t_0)$ and $\mathbf{H}_{clock}$ do not commute (and cannot commute if the clock is to operate properly), the smaller the uncertainty in $y(t_0)$, the greater the uncertainty in H_{clock}. We will see that if we want to use this clock to measure the time of an event, then we will encounter the limitation given by (3.39). We will need to ensure that initially the position of the clock is uncertain in order for our measurements of the time of an event to succeed.

The reason for this is that since $\mathbf{y}$ is conjugate to $\mathbf{H}_{clock} = \mathbf{P_y}$, accurate clocks (which are narrow in y) have a large spread in P_y. This means that in general the energy of an accurate clock can take on fairly large values. For an infinitely accurate clock the energy will almost always be infinite. Accurate clocks therefore, have a large energy uncertainty, and this makes them very hard to use to measure the time of an event. This is because accurate clocks are usually so energetic that they need a large amount of energy to turn them off. To measure the time-of-arrival of a particle, the particle itself will have to turn off the clock when it arrives – the external observer cannot supply any energy since she does not know when to turn the clock off. If the clock is much more energetic than the particle, then it will be impossible for the particle to turn off the clock and no record will exist that the particle arrived. In fact, the particle may be reflected and will never arrive. The situation is very different from usual measurements, where an external observer supplies the energy to make the measurement. If we were to measure the time of the clock, we would have to supply a large amount of energy to make an accurate measurement. However, in order to measure the time-of-arrival, it is the particle which must supply the energy to stop the clock. To see this, let us use the clock to measure the time-of-arrival.

3.3 Measurement of Time-of-Arrival

In this section we consider toy models of a measurement of time-of-arrival. To begin with, assume that the particle interacts with a detector that is located at $x = 0$ and is coupled to a clock. Initially, as the particle is prepared, the clock is set to show $t = 0$. Our purpose is to design a particular set-up such that as a particle crosses the point $x = 0$ the detector stops the clock. Since the masses of the particle detector and the clock are unlimited we can ignore the uncertainty in the position of the measurement device and assume it is properly positioned at $x = 0$. We shall consider four models. The first model describes a direct interaction of the particle with the clock. In the second model, the particle is detected by a two-level detector, which turns the clock off. To avoid the reflection due to the clock's energy, we look next at the possibility of boosting the energy of the particle in order to turn off the clock. We shall also consider the case of a "smeared" interaction, and conclude with a general discussion.

3.3.1 Measurement with a clock

The simplest model which describes a direct interaction of a particle and a clock [16], without additional "detector" degrees of freedom, is described by the Hamiltonian

$$H = \frac{1}{2m}\mathbf{P_x}^2 + \theta(-\mathbf{x})\mathbf{P_y}. \tag{3.44}$$

Here, the particle's motion is confined to one spatial dimension, x, and $\theta(x)$ is a step function. The clock's Hamiltonian is represented by $\mathbf{P_y}$, and the time is recorded on the conjugate variable $\mathbf{y}$. Because of the step function, the clock will stop when the particle is located to the right of the origin.

The equations of motion read:

$$\dot{\mathbf{x}} = \mathbf{P_x}/m, \qquad \dot{\mathbf{P}}_\mathbf{x} = -\mathbf{P_y}\delta(\mathbf{x}) \tag{3.45}$$

$$\dot{\mathbf{y}} = \theta(\mathbf{x}), \qquad \dot{\mathbf{P}}_\mathbf{y} = 0. \tag{3.46}$$

At $t \to \infty$ the clock shows the time of arrival:

$$\mathbf{y}_\infty = \mathbf{y}(t_0) + \int_{t_0}^{\infty} \theta(-\mathbf{x}(t))dt \tag{3.47}$$

A crucial difference between the classical and the quantum case, can be noted from Equation (3.45). In the classical case the back-reaction can be made negligible small by choosing $P_y \to 0$. In this case, the particle follows the undisturbed solution, $x(t) = x(t_0) + \frac{p_x}{m}(t - t_0)$. If initial we set $y(t_0) = t_0$ and $x(t_0) < 0$ the clock finally reads:

$$y_\infty = y(t_0) + \int_{t_0}^{\infty} \theta[-x(t_0) - \frac{p_x}{m}(t - t_0)]dt = -\frac{mx(t_0)}{p_x}. \tag{3.48}$$

The classical time-of-arrival is $t_A = y_\infty = -mx(t_0)/p_x$. The same result would have been obtained by measuring the classical variable $-mx_0/p_x = -mx(t)/p_x + (t - t_0)$, at arbitrary time t. Consequently, the continuous and the indirect measurements alluded to in Section 3.1, are classically equivalent.

On the other hand, in quantum mechanics the uncertainty relation dictates a strong back-reaction, i.e. in the limit of $\Delta y = \Delta t_A \to 0$, p_y in (3.45) must have a large uncertainty, and the state of the particle must be strongly affected by the act of measuring. Therefore, the two classically equivalent measurements become inequivalent in quantum mechanics.

Before we proceed to examine the continuous measurement process in more detail, we note that a more symmetric formulation of the above measurement exists in which knowledge of the direction from which particles are arriving is not needed. We can consider

$$H = \frac{1}{2m}\mathbf{P_x}^2 + \theta(-\mathbf{x})\mathbf{P_{y_1}} + \theta(\mathbf{x})\mathbf{P_{y_2}}. \tag{3.49}$$

As before, the particle's motion is confined to one spatial dimension, x. Two clocks are represented by $\mathbf{P_{y_1}}$ and $\mathbf{P_{y_2}}$, and time is recorded on the conjugate variables $\mathbf{y}_1$ and $\mathbf{y}_2$, respectively.

The first clock operates only when the particle is located at $x < 0$ and the second clock at $x > 0$. For example, if we start with a beam of particle at $x < 0$, a measurement at $t \to \infty$ of $\mathbf{y}_1$ gives the time-of-arrival. Alternatively we could measure $t - \mathbf{y}_2$. As a check we have $\mathbf{y}_1 + \mathbf{y}_2 = t$. It is harder to determine the time-of-arrival if the particle arrives from both directions. If however it is known that initially $|x| < L$, we can measure $\mathbf{y}_1$ and $\mathbf{y}_2$ after $t >> L/v$. The time-of-arrival will then be given by $t_A = \min(\mathbf{y}_1, \mathbf{y}_2)$.

For simplicity we shall examine in more detail the case of only one clock and a particle initially at $x < 0$, which travels towards the clock at $x = 0$. The eigenstates of the Hamiltonian are

$$\phi_{kp}(x,y,t) = \begin{cases} (e^{ikx} + A_R e^{-ikx})e^{ipy - i\omega(t)} & x < 0 \\ A_T e^{iqx + ipy - i\omega(t)} & x \geq 0 \end{cases} \tag{3.50}$$

where k and p, are the momentum of the particle and the clock, respectively, and $\omega(t) =$

$\frac{k^2t}{2m} + pt$. Continuity of ϕ_{kp} requires that

$$\begin{aligned} A_T &= \frac{2k}{k+q} \\ A_R &= \frac{k-q}{k+q}, \end{aligned} \tag{3.51}$$

where $q = \sqrt{k^2 + 2mp} = \sqrt{2m(E_k + p)}$.

The solution of the Schrödinger equation is

$$\psi(x,y,t) = N \int_{-\infty}^{\infty} dk \int_0^{\infty} dp f(p) g(k) \phi_{kp}(x,y,t), \tag{3.52}$$

where N is a normalization constant and $f(p)$ and $g(k)$ are some distributions. For example, with

$$\begin{aligned} f(p) &= e^{-\Delta_y{}^2(p-p_0)^2} \\ g(k) &= e^{-\Delta_x{}^2(k-k_o)^2 + ikx_0}. \end{aligned} \tag{3.53}$$

and $x_0 > 0$, the particle is initially localized on the left ($x < 0$) and the clock (with probability close to 1) runs. The normalization in eq. (3.52) is thus $N^2 = \frac{\Delta x \Delta y}{2\pi^3}$. By choosing $p_0 \approx 1/\Delta_y$, we can now set the the clock's energy in the range $0 < p < 2/\Delta_y$.

Let us first show that in the stationary point approximation the clock's final wave function is indeed centered around the classical time-of-arrival. Thus we assume that Δ_y and Δ_x are large such that $f(p)$ and $g(k)$ are sufficiently peaked. For $x > 0$, the integrand in (3.52) has an imaginary phase

$$\theta = qx + kx_o + py - \frac{k^2t}{2m} - pt. \tag{3.54}$$

$\frac{d\theta}{dk} = 0$ implies

$$x_{peak}(p) = -\frac{q(k_0)}{k_0} x_o + \frac{q(k_0)t}{m}, \tag{3.55}$$

and $\frac{d\theta}{dp} = 0$ gives

$$y_{peak}(k) = t - \frac{mx}{q_0}. \tag{3.56}$$

Hence at $x = x_{peak}$ the clock coordinate y is peaked at the classical time-of-arrival

$$y = \frac{mx_o}{k_0}. \tag{3.57}$$

To see that the clock yields a reasonable record of the time-of-arrival, let us consider further the probability distribution of the clock

$$\rho(y, y)_{x>0} = \int dx |\psi(x > 0, y, t)|^2. \tag{3.58}$$

In the case of inaccurate measurements with a small back-reaction on the particle $A_T \simeq 1$. The clocks density matrix is then found (see Appendix B) to be given by:

$$\rho(y, y)_{>0} \simeq \frac{1}{\sqrt{2\pi\gamma(y)}} e^{-\frac{(y-t_c)^2}{2\gamma(y)}} \tag{3.59}$$

where the width is $\gamma(y) = \Delta y^2 + (\frac{m\Delta x}{k_o})^2 + (\frac{y}{2k_o\Delta x})^2$. As expected, the distribution is centered around the classical time-of-arrival $t_c = x_o m/k_o$. The spread in y has a term due to the initial width Δy in clock position y. The second and third term in $\gamma(y)$ is due to the kinematic spread in the time-of-arrival $\frac{1}{dE} = \frac{m}{kdk}$ and is given by $\frac{dx(y)\, m}{k_o}$ where $dx(y)^2 = \Delta x^2 + (\frac{y}{2m\Delta x})^2$. The y dependence in the width in x arises because the wave function is spreading as time increases, so that at later y, the wave packet is wider. As a result, the distribution differs slightly from a Gaussian although this effect is suppressed for particles with larger mass.

When the back-reaction causes a small disturbance to the particle, the clock records the time-of-arrival. What happens when we wish to make more accurate measurements? Consider the exact transition probability $T = \frac{q}{k}|A_T|^2$, which also determines the probability to stop the clock. The latter is given by

$$T = \sqrt{\frac{E_k + p}{E_k}} \left[\frac{2\sqrt{E_k}}{\sqrt{E_k} + \sqrt{E_k + p}} \right]^2. \tag{3.60}$$

Since the possible values obtained by p are of the order $1/\Delta_y \equiv 1/\Delta t_A$, the probability to trigger the clock remains of order one only if

$$\bar{E}_k \delta t_A > 1. \qquad (3.61)$$

Here δt_A stands for the initial uncertainty in position of the dial $\mathbf{y}$ of the clock, and is interpreted as the accuracy of the clock. $\bar{E}_k$ can be taken as the typical initial kinetic energy of the particle.

In measurements with accuracy better then $1/\bar{E}_k$ the probability to succeed drops to zero like $\sqrt{E_k \delta t_A}$, and the time-of-arrival of most of the particles cannot be detected. Furthermore, the probability distribution of the fraction which has been detected depends on the accuracy δt_A and can become distorted with increased accuracy. This observation becomes apparent in the following simple example. Consider an initial wave packet that is composed of a superposition of two Gaussians centered around $k = k_1$ and $k = k_2 >> k_1$. Let the classical time-of-arrival of the two Gaussians be t_1 and t_2 respectively. When the inequality (3.61) is satisfied, two peaks around t_1 and t_2 will show up in the final probability distribution. On the other hand, for $\frac{2m}{k_1^2} > \delta t_A > \frac{2m}{k_2^2}$, the time-of-arrival of the less energetic peak will contribute less to the distribution in y, because it is less likely to trigger the clock. Thus, the peak at t_1 will be suppressed. Clearly, when the precision is finer than $1/\bar{E}_k$ we shall obtain a distribution which is considerably different from that obtained for the case $\delta t_A > 1/\bar{E}_k$ when the two peaks contribute equally.

3.3.2 Two-level detector with a clock

A more realistic set-up for a time-of-arrival measurement is one that also includes a particle detector which switches the clock off as the particle arrives. We shall describe the particle detector as a two-level spin degree of freedom. The particle will flip the state of the trigger from "on" to "off", i.e.. from $\uparrow_z$ to $\downarrow_z$. First let us consider a model for the trigger without including the clock:

$$H_{trigger} = \frac{1}{2m}\mathbf{P}_{\mathbf{x}}^{\mathbf{2}} + \frac{\alpha}{2}(1+\sigma_x)\delta(\mathbf{x}). \tag{3.62}$$

The particle interacts with the repulsive Dirac delta function potential at $x = 0$, only if the spin is in the $|\uparrow_x\rangle$ state, or with a vanishing potential if the state is $|\downarrow_x\rangle$. In the limit $\alpha \to \infty$ the potential becomes totally reflective (Alternatively, one could have considered a barrier of height α^2 and width $1/\alpha$.) In this limit, consider a state of an incoming particle and the trigger in the "on" state: $|\psi\rangle|\uparrow_z\rangle$. This state evolves to

$$|\psi\rangle|\uparrow_z\rangle \quad \to \quad \frac{1}{\sqrt{2}}\Big[|\psi_R\rangle|\uparrow_x\rangle + |\psi_T\rangle|\downarrow_x\rangle\Big], \tag{3.63}$$

where ψ_R and ψ_T are the reflected and transmitted wave functions of the particle, respectively.

The latter equation can be rewritten as

$$\frac{1}{2}|\uparrow_z\rangle(|\psi_R\rangle + |\psi_T\rangle) + \frac{1}{2}|\downarrow_z\rangle(|\psi_R\rangle - |\psi_T\rangle) \tag{3.64}$$

Since $\uparrow_z$ denotes the "on" state of the trigger, and $\downarrow_z$ denotes the "off" state, we have flipped the trigger from the "on" state to the "off" state with probability 1/2 [2]. Although this model only works half the time, the chance of success does not depend in any way on the system, and in particular, on the particle's energy. Furthermore, one can construct models where a detector is triggered almost all the time [35], although with some energy dependence in the probability of triggering.

So far we have succeeded in recording the event of arrival to a point. We have no information at all on the time-of-arrival. It is also worth noting that the net energy exchange between the trigger and the particle is zero, i.e.. the particle's energy is unchanged.

[2]It is interesting to see that for some wave functions which represent coherent superpositions of particles arriving from both the left and right, the detector is never triggered. An example of this is given in Appendix A.

This model leads us to reject the arguments of Allcock. He considers a detector which is represented by a pure imaginary absorber $H_{int} = iV\theta(-\mathbf{x})$. Allcock's claim is that measuring the time-of-arrival is equivalent to absorbing a particle in a finite region. If you can absorb the particle in an arbitrarily short time, then you have succeeded in transferring the particle from an incident channel into a detector channel and the time-of-arrival can then be recorded. Using his interaction Hamiltonian one finds that the particle is absorbed in a rate proportional to V^{-1}. One can increases the rate of absorption by increasing V, but the particle will be reflected unless $V << E_k$. He therefore claims that since you cannot absorb the particle in an arbitrarily short time, you cannot record the time-of-arrival with arbitrary accuracy.

However, our two level detector is equivalent to a detector which absorbs a particle in an arbitrarily short period of time, and then transfers the information to another channel. The particle is instantaneously converted from one kind of particle (spin up), to another kind of particle (spin down). We therefore see that considerations of absorption alone do not place any restrictions on measuring the time-of-arrival.

However, we shall see that when we proceed to couple the trigger to a clock we do find a limitation on the time-of-arrival. A model for this coupling can be given by the Hamiltonian

$$H_{trigger+clock} = \frac{1}{2m}\mathbf{P}_{\mathbf{x}}^{\mathbf{2}} + \frac{\alpha}{2}(1+\sigma_x)\delta(\mathbf{x}) + \frac{1}{2}(1+\sigma_z)\mathbf{P}_{\mathbf{y}}. \tag{3.65}$$

Since we can have $\alpha >> P_y$ it would seem that the triggering mechanism need not be affected by the clock. If the final wave function includes a non-vanishing amplitude of $\downarrow_z$, the clock will be turned off and the time-of-arrival recorded. However, the exact solution shows that this is not the case. Consider for example an initial state of an incoming wave from the left and the spin in the $\uparrow_z$ state.

The eigenstates of the Hamiltonian in the basis of σ_z are

$$\Psi_L(x) = \begin{pmatrix} e^{ik_\uparrow x} + \phi_{L\uparrow} e^{-ik_\uparrow x} \\ \phi_{L\downarrow} e^{-ik_\downarrow x} \end{pmatrix} e^{ipy}, \tag{3.66}$$

for $x < 0$ and

$$\Psi_R(x) = \begin{pmatrix} \phi_{R\uparrow} e^{ik_\uparrow x} \\ \phi_{R\downarrow} e^{ik_\downarrow x} \end{pmatrix} e^{ipy}, \tag{3.67}$$

for $x > 0$. Here $k_\uparrow = \sqrt{2m(E-p)} = \sqrt{2mE_k}$ and $k_\downarrow = \sqrt{2mE} = \sqrt{2m(E_k + p)}$.

Matching conditions at $x = 0$ yields

$$\phi_{R\uparrow} = \frac{\frac{2k_\uparrow}{m\alpha} - \frac{k_\uparrow}{k_\downarrow}}{\frac{2k_\uparrow}{m\alpha} - (1 + \frac{k_\uparrow}{k_\downarrow})} \tag{3.68}$$

$$\phi_{R\downarrow} = \frac{k_\uparrow}{k_\downarrow}((\phi_{R\uparrow} - 1) = \frac{\frac{k_\uparrow}{k_\downarrow}}{\frac{2k_\uparrow}{m\alpha} - (1 + \frac{k_\uparrow}{k_\downarrow})}, \tag{3.69}$$

and

$$\phi_{L\downarrow} = \phi_{R\downarrow} \tag{3.70}$$

$$\phi_{L\uparrow} = \phi_{R\uparrow} - 1. \tag{3.71}$$

We find that in the limit $\alpha \to \infty$ the transmitted amplitude is

$$\phi_{R\downarrow} = -\phi_{R\uparrow} = \frac{\sqrt{E_k}}{\sqrt{E_k} + \sqrt{E_k + p}}. \tag{3.72}$$

Precisely as in the previous section, the transition probability decays like $\sqrt{E_k/p}$. From eqs. (3.70,3.71) we get that $\phi_{L\downarrow} \to 0$, and $\phi_{L\uparrow} \to 1$ as the accuracy of the clock increases. Hence the particle is mostly reflected back and the spin remains in the $\uparrow_z$ state; i.e., the clock remains in the "on" state.

The present model gives rise to the same difficulty as the previous model. Without the clock, we can flip the "trigger" spin by means of a localized interaction, but when we couple the particle to the clock, the probability to flip the spin and turn the clock off decreases gradually to zero as the clock's precision is improved.

3.3.3 Local amplification of kinetic energy

The difficultly with the previous examples seems to be that the particle's kinetic energy is not sufficiently large, and energy can not be exchanged with the clock. To overcome this difficulty one can imagine introducing a "pre-booster" device just before the particle arrives at the clock. If it could boost the particle's kinetic energy to an arbitrarily high value, without distorting the incoming probability distribution (i.e. amplifying all wave components k equally), and at an arbitrary short distance from the clock, then the time-of-arrival could be measured to arbitrary accuracy. Thus, an equivalent problem is: can we boost the energy of a particle by using only localized (time independent) interactions?

Let us consider the following toy model of an energy booster described by the Hamiltonian

$$H = \frac{1}{2m}\mathbf{P}_{\mathbf{x}}^{\mathbf{2}} + \alpha\sigma_x\delta(\mathbf{x}) + \frac{W}{2}\theta(\mathbf{x})(1+\sigma_z) + \frac{1}{2}[V_1\theta(-\mathbf{x}) - V_2\theta(\mathbf{x})](1-\sigma_z). \qquad (3.73)$$

Here, $\alpha, W,$ V_1 and V_2 are positive constants. Let us consider an incoming wave packet propagating from left to right. The role of the term $\alpha\sigma_x\delta(x)$ is to flip the spin $\uparrow_z$ to $\downarrow_z$. The V_2 term is the booster, and particles which cross into the region $x > 0$ will be boosted in kinetic energy by the amount V_2. The other parts of the potential serve to damp out undesirable components of the wave function which can interfere with each other if a clock is placed close to the origin. The $\uparrow_z$ component of the wave function is damped out exponential by the W term for $x > 0$. The $\downarrow_z$ component is damped out for $x < 0$ by the term V_1. As we shall see, for a given momentum k, one can chose the four free parameters above such that the wave is transmitted through the booster with probability 1, while the gain in energy V_2 can be made arbitrarily large. The potential barrier W can also be made arbitrarily large. The last requirement means that the unflipped component decays for $x > 0$ on arbitrary short scales, which allows us to locate the booster arbitrarily close to the clock, while preventing destructive interference between the flipped and un-flipped transmitted waves.

The eigenstates of (3.73), in the basis of σ_z, are given by

$$\Psi_L(x) = \begin{pmatrix} e^{ikx} + \phi_{L\uparrow} e^{-ikx} \\ \phi_{L\downarrow} e^{qx} \end{pmatrix} \tag{3.74}$$

for $x < 0$ and

$$\Psi_R(x) = \begin{pmatrix} \phi_{R\uparrow} e^{-\lambda x} \\ \phi_{R\downarrow} e^{ik' x} \end{pmatrix} \tag{3.75}$$

for $x > 0$, where $k^2 = V_1 - q^2 = -\lambda^2 + W = -V_2 + k'^2$. Matching conditions at $x = 0$ we find

$$\phi_{L\uparrow} = \phi_{R\uparrow} - 1 = \frac{k'k + q\lambda + i(kq - k'\lambda) - \alpha^2}{k'k - q\lambda + i(k'\lambda + kq) + \alpha^2}, \tag{3.76}$$

$$\phi_{R\downarrow} = \phi_{L\downarrow} = \frac{\alpha}{ik' - q}\Big(1 + \phi_{L\uparrow}\Big). \tag{3.77}$$

For a given k, W and V_2 (or given $k,\ \lambda$ and k') we still are free to chose α and V_1 (or q). We now demand that

$$\alpha = k'k + q\lambda, \qquad q = \lambda\frac{k'}{k}. \tag{3.78}$$

With this choice we obtain for the transmission and reflection probabilities:

$$R_\uparrow = 0, \qquad T_\downarrow = \frac{k'}{k}|\phi_{R\downarrow}|^2 = 1. \tag{3.79}$$

Therefore, the wave has been fully transmitted and the spin has flipped with probability 1.

So far we have considered an incoming wave with fixed momentum k. For a general incoming wave packet only a part of the wave will be transmitted and amplified. Furthermore one can verify that the amplified transmitted wave has a different form than the original wave function since different momenta have different probabilities of being amplified. Thus, in general, although amplification is possible and indeed will lead to a much higher rate of detection, it will give rise to a distorted probability distribution for the time-of-arrival.

There is however one limiting case in which the method does seem to succeed. Consider a narrow wave peaked around k with a width dk. To first order in dk, the probability $T_\downarrow$ that the particle is successfully boosted is given by

$$T_\downarrow \simeq 1 - \frac{2dk}{k}. \tag{3.80}$$

Therefore in the special case that $\frac{dk}{k} << 1$, the transition probability is still close to one. If in this case we known in advance the value of k up to $dk << k$, we can indeed use the booster to improve the bound (3.61) on the accuracy.

The reason why this seems to work in this limiting case is as follows. The probability of flipping the particle's spin depends on how long it spends in the magnetic field described by the α term in (3.73). If however, we know beforehand, how long the particle will be in this field, then we can tune the strength of the magnetic field (α) so that the spin gets flipped. The requirement that $dk/k << 1$ is thus equivalent to having a small uncertainty in the "interaction time" with this field. In some sense, the measurement is possible, because we know the particle's momentum before hand. Of course, if we have prior knowledge of the particle's momentum, then we could just measure $\mathbf{x}$ and argue that this allows us to calculate the time of arrival through $t_A = mx/p$. We therefore believe that the reason the measurement procedure described above works in this limiting case is because it assumes prior knowledge of the particle's momentum, and we do not believe that one could improve it so that it works for all states. These "booster" measurements cannot be used for general wave functions, and even in the special case above, one still requires some prior information of the incoming wave function.

3.3.4 Gradual triggering of the clock

In order to avoid the reflection found in the previous two models, we shall now replace the sharp step-function interaction between the clock and particle by a more gradual transition.

When the WKB condition is satisfied

$$\frac{d\lambda(x)}{dx} = \epsilon << 1 \tag{3.81}$$

where $\lambda(x)^{-2} = 2m[E_0 - V(x)]$, the reflection amplitude vanishes as

$$\sim \exp(-1/\epsilon^2) \tag{3.82}$$

Solving the equation for the potential with a given ϵ we obtain

$$V_\epsilon(x) = E_0 - \frac{1}{2m\epsilon^2}\frac{1}{x^2} \tag{3.83}$$

Now we observe that any particle with $E \geq E_0$ also satisfies the WKB condition (3.81) above for the *same* potential V_ϵ. Furthermore $p_y V_\epsilon$ also satisfies the condition for any $p_y > 1$.

These considerations suggest that we should replace the Hamiltonian in eq. (3.49) with

$$H = \mathbf{P_x}^2/2m + V(x)\mathbf{P_y} \tag{3.84}$$

where

$$V(x) = \begin{cases} -\frac{x_A^2}{x^2} & x < x_A \\ -1 & x \geq x_A \end{cases} \tag{3.85}$$

Here $x_A^{-2} = 2m\epsilon^2$.

Thus this model describes a gradual triggering *on* of the clock which takes place when the particles propagates from $x \to -\infty$ towards $x = x_A$. In this case the arrival time is approximately given by $t - \mathbf{y}$ where $t = t_f - t_i$. Since without limiting the accuracy of the clock we can demand that $p_y >> 1$, the reflection amplitude off the potential step is exponentially small for *any* initial kinetic energy E_k.

The problem is however that the final value of $t - \mathbf{y}$ does not always correspond to the time-of-arrival since it contains errors due to the affect of the potential $V(x)$ on the particle which we shall now proceed to examine.

In the following we shall ignore ordering problems and solve for the classical equations of motion for (3.84). We have

$$y(t_f) - y(t_i) = \int_{t_i}^{t_f} V(\mathbf{x}(t'))dt' \tag{3.86}$$

which can be decomposed to

$$y(t_f) - y(t_i) = (t_i - t_0) + (t_f - t_i) + \int_{t_i}^{t_0} V(x(t'))dt' \equiv A + B + C \tag{3.87}$$

where

$$A = \frac{1}{\sqrt{2mE}}\left[\sqrt{x_A^2 + p_y x_A^2/E} - \sqrt{x_i^2 + p_y x_A^2/E}\right] \tag{3.88}$$

is the time that the particle travels from x_i to x_A in the potential $p_y V(x)$, B is the total time, and

$$C = -\frac{x_A}{\sqrt{2mp_y}}\left[\log \frac{1 + \sqrt{1 + \frac{E}{p_y}}}{1 + \sqrt{1 + \frac{Ex_i^2}{p_y x_A^2}}} + \log \frac{x_i}{x_A}\right] \tag{3.89}$$

The last term C, corresponds to an error due to the imperfection of the clock, i.e. the motion of the clock prior to arrival to x_A. By making p_y large we can minimize the error from this term to $\sim (x_A \log p_y / \sqrt{2mp_y})$.

Inspecting equation (3.87) we see that by measuring $y_f - y_i$ and then subtracting $B = t_f - t_i$ (which is measured by another clock) we can determine the time $t_0 - t_i$, which is the time-of-arrival for a particle in a potential $p_y V(x)$, up to the correction C. However this time reflects the motion in the presence of an external (unknown) potential, while we are interested in the time-of-arrival for a free particle.

Nevertheless, if $p_y/E << 1$ we obtain

$$-A = \frac{x_A - x_i}{\sqrt{2mE}} + O\left(\frac{p_y}{E}\right) \tag{3.90}$$

The time-of-arrival can hence be measured provided that $E_k\delta t >> 1$. On the other hand, when the detector's accuracy is $\delta t < 1/E$, the particle still triggers the clock. However the measured quantity, A, no longer correspond to the time-of-arrival. Again, we see that when we ask for too much accuracy, the particle is strongly disturbed and reading of the clock has nothing to do with the time-of-arrival of a free particle.

3.3.5 General considerations

We have examined several models for a measurement of time-of-arrival and found a limitation,

$$\delta t_A > 1/\bar{E}_k, \tag{3.91}$$

on the accuracy that t_A can be measured. Is this limitation a general feature of quantum mechanics?

First we should notice that eq. (3.91) does not seem to follow from the uncertainty principle. Unlike the uncertainty principle, whose origin is kinematic, (3.91) follows from the nature of the *dynamic* evolution of the system during a measurement. Furthermore here we are considering a restriction on the accuracy (not uncertainty) of a single measurement. While it is difficult to provide a general proof, in the following we shall indicate why (3.91) is expected to hold under more general circumstances.

Let us examine the basic features that gave rise to (3.91). In the toy models considered in Sections 3.3.1 and 3.3.2, the clock and the particle had to exchange energy $p_y \sim 1/\delta t_A$. As a result, the effective interaction by which the clock switches off, looks from the point of view of the particle like a step function potential. This led to "non-detection" when (3.91) was violated.

Can we avoid this energy exchange between the particle and the clock? Let us try to deliver this energy to some other system without modifying the energy of the particle.

For example consider the following Hamiltonian for a clock with a reservoir:

$$H = \frac{{\mathbf{P_x}}^2}{2m} + \theta(-\mathbf{x})H_c + H_{res} + V_{res}\theta(\mathbf{x}) \tag{3.92}$$

The idea is that when the clock stops, it dumps its energy into the reservoir, which may include many other degrees of freedom, instead of delivering it to the particle. In this model, the particle is coupled directly to the clock and reservoir, however we could as well use the idea of Section 3.3.2 above. In this case:

$$H = \frac{{\mathbf{P_x}}^2}{2m} + \frac{\alpha}{2}(1+\sigma_x)\delta(\mathbf{x}) + \frac{1}{2}(1+\sigma_z)H_c + H_{res} + \frac{1}{2}(1-\sigma_z)V_{res}. \tag{3.93}$$

The particle detector has the role of providing a coupling between the clock and reservoir.

Now we notice that in order to transfer the clock's energy to the reservoir without affecting the free particle, we must also prepare the clock and reservoir in an initial state that satisfies the condition

$$H_c - V_{res} = 0 \tag{3.94}$$

However this condition does not commute with the clock time variable $\mathbf{y}$. We can measure initially $\mathbf{y} - \mathbf{R}$, where R is a collective degree of freedom of the reservoir such that $[\mathbf{R}, V_{res}] = i$, but in this case we shall not gain information on the time-of-arrival y since R is unknown. We therefore see that in the case of a sharp transition, i.e. for a localized interaction with the particle, one cannot avoid a shift in the particle's energy. The "non-triggering" (or reflection) effect cannot be avoided.

We have also seen that the idea of boosting the particle "just before" it reaches the detector, fails in the general case. What happens in this case is that while the detection rate increase, one generally destroys the initial information stored in the incoming wave packet. Thus although higher accuracy measurements are now possible, they do not reflect directly the time-of-arrival of the initial wave packet.

Finally we note that in reality, measurements usually involve some type of cascade effect, which lead to signal amplification and finally allows a macroscopic clock to be triggered. A typical example of this type would be the photo-multiplier where an initially small energy is amplified gradually and finally detected. Precisely this type of process occurs also in the model of section 3.3.4. In this case the particle gains energy gradually by "rolling down" a smooth step function. It hence always triggers the clock. The basic problem with such a detector is that when (3.91) is violated, the "back reaction" of the detector on the particle, during the gradual detection, becomes large. The relation between the final record to the quantity we wanted to measure is lost.

We have examined various models for the measurement of time-of-arrival, t_A, and found a basic limitation on the accuracy that t_A can be determined reliably: $\delta t_A > 1/\bar{E}_k$. This limitation is quit different in origin from that due to the uncertainty principle; here it applies to a *single* quantity. Furthermore, unlike the kinematic nature of the uncertainty principle, in our case the limitation is essentially dynamical in its origin; it arises when the time-of-arrival is measured by means of a continuous interaction between the measuring device and the particle.