2.1 Probabilities at a Time and in Time

Within quantum mechanics, a complete set of commuting observables can be found which describe the attributes of a system at a given time. However, difficulties arise for attributes of a system that extend over time, such as the time of an atomic decay, the time of arrival, etc. As we discuss below, simple extensions of ordinary notions of probabilities *at a certain time* to probabilities *in time* give rise to distributions which can no longer be interpreted as probabilities. The reason for this can be understood in simple terms. Consider for example, the event of a particle entering a box. What is the time of the event? Classically, there is no distinction between attributes at one time or in time. One can, for example, measure the position and momentum of the particle at any time with negligible disturbance and use this information to deduce the time of entering the box. Quantum mechanically, however, there are two separate questions. We can either ask at a certain time t_0, "has the particle already entered the box?", or we can ask "when did the particle enter the box?" To answer the first question we simply measure at time t_0 if the particle is in the box. Although quantum mechanics does provides us with a prediction for the probability $P(t_0)$ for this event, this probability does not describe a probability in time. The measurement at time t_0 disturbs the evolution of the system in the future and hence the probability distribution at time $t > t_0$ will no longer be given by $P(t)$. In fact by the same consideration, we see that the two questions above, or the corresponding measurements, are not compatible with each other, in the sense that one measurement disturbs the other and visa-versa. One would think that the questions "when did an event occur?" and "has the event occurred?" can be answered simultaneously, however, as we shall see, they are in fact complementary. One cannot necessarily answer both questions simultaneously. In the next section we will formulated these difficulties in a more precise way.

We will then examine two specific cases of measuring the time of an event. One is the arrival of a particle to a certain location, and another is a recent proposal of Rovelli [28] to measure the time that a measurement occurred. We argue that his scheme only answers the first question: "has the measurement occurred already at a certain time?", but does not answer the more difficult question "when did the measurement occur?" In other words, it does not provide a proper probability distribution for the time of an event. We also discuss the relationship between Rovelli's measurement scheme, and the use of the probability current for measurements of time-of-arrival. In Section 2.4 we discuss a model and set of operators which can be used to determine a probability distribution for the time of an event. The model is based upon a continuous process akin to a rapid series of measurements. We find that in the limit of high accuracy the system is severely disturbed and the measurement does not work, an effect which is analogous to the Zeno paradox.

2.2 Did it Occur vs. When Did it Occur

In conventional quantum mechanics, for each observable, we can assign a set of projection operators $\mathbf{\Pi}_i$ onto a set of eigenstates ϕ_i of some operator. At each time t there exists a Hilbert space and inner product which enable one to calculate the probability $P_i(t)$ that the system is in one of the states ϕ_i. In certain cases, one can find a subset a of the set i such that the projection operator

$$\mathbf{\Pi}_a = \sum_{i \in a} \mathbf{\Pi}_i \tag{2.5}$$

gives the probability that a certain event a has happened. For example, $\mathbf{\Pi}_a$ may project onto the set of states of an atom which has decayed into its ground state and emitted a photon. Or, in one of the examples which we will be discussing in Section 2.3, $\mathbf{\Pi}_a$ will be the projector onto the states of a measuring device after a measurement has occurred.

For the case of time-of-arrival, $\mathbf{\Pi}_a$ will be the projector onto a region of the x-axis (in this case, the index i is continuous).

If initially the system is in the state ψ then in the Heisenberg representation the probability that the event has happened at any time t is given by

$$P_a(t) = \langle\psi|\mathbf{\Pi}_a(t)|\psi\rangle \ . \tag{2.6}$$

One can also compute the "current operator"

$$\mathbf{J}_a = \frac{d\mathbf{\Pi}_a(t)}{dt} \tag{2.7}$$

which gives the rate of change of the probability distribution $P_a(t)$. It is tempting to argue that the probability distribution

$$p_a(t) = \langle\psi|\mathbf{J}_a(t)|\psi\rangle \tag{2.8}$$

gives the probability that the event a happens between t and dt, since classically the probability that an event happened some time before time t is just the integral between some initial time t_o and t of the probability that the event happens at that time.

However, the probability distribution obtained from $\mathbf{J}_a$ cannot be thought of as a probability distribution *in time*. $p_a(t)$ is not the probability that the event happened at time t. To see that $p_a(t)$ is not a probability distribution in time, let us compare its properties to the properties (1-4) of the conventional quantum [1] probability distribution obtained from the projectors $\mathbf{\Pi}_i$.

Property 1 *The probability of finding that the system is in the state ϕ_i at time t is independent of the probability of finding that the system is in the state ϕ_j (at the same time t).*

[1] properties 2-4 are also true of classical probability distributions

i.e..

$$[\mathbf{\Pi}_i(t), \mathbf{\Pi}_j(t)] = 0. \qquad (2.9)$$

If we interpret the probabilities $P_a(t)$ as probabilities in time, then our conventional notions of what these probabilities mean, break down. In general,

$$[\mathbf{\Pi}_a(t), \mathbf{\Pi}_a(t')] \neq 0. \qquad (2.10)$$

Measurements made at earlier times influence measurements made at later times. The possible results of an observable at time t will depend on whether there were any previous measurements of that observable. In classical mechanics, one can make the interaction of the measuring device with the system arbitrarily weak, and therefore, not disturb the evolution of the system in time, but this is not true in quantum mechanics. A measurement of position at t_1 for example, will disturb the momentum of the particle in such a way that future measurements of position at t_2 will give very different results from the case where no measurement was performed at t_1. Since $\mathbf{\Pi}_a(t)$ does not commute with itself at different times, there is no reason to believe that $\mathbf{J}_a$ will commute with itself at different times either. It is essentially this difference between conventional probabilities and those obtained from $\mathbf{\Pi}_a$ which prevents us from determining when an event occurred.

In addition, $p_a(t)$ and $P_a(t)$ do not have the following other properties of quantum distributions:

Property 2 *If $i \neq j$ then the projection operators project onto orthogonal states.*

i.e..

$$\mathbf{\Pi}_i(t)\mathbf{\Pi}_j(t) = 0 \quad i \neq j \ . \qquad (2.11)$$

For example, if a particle is found at position x, then it could not have been anywhere else at the same time. On the other hand, a particle may be at the same position at many different times. There is no reason why the event a can not happen at many different times. In general

$$\mathbf{\Pi}_a(t)\mathbf{\Pi}_a(t') \neq 0. \tag{2.12}$$

This is also true of classical distributions. For example, I can only be at one place at one time, but I can be at that same place at many different times.

Property 3 *The probabilities $P_i(t)$, are normalized at a given time.*

i.e..

$$\sum_i P_i(t) = 1 \tag{2.13}$$

However, the operator $\mathbf{J}_a$ is not necessarily normalized in time

$$\int_{-\infty}^{\infty} dt \frac{dP_a(t)}{dt} = \lim_{t->\infty} P_a(t) - P_a(-t). \tag{2.14}$$

In special physical circumstances, $P_a(t)$ may initially be zero, and may finally equal one in the distant future, but there is no reason to expect this to be true in general. One might try to renormalize $\mathbf{J}_a$, but for each initial state the normalization will in general be different.

This property is also true of classical probability distributions. They also must be normalized, and the classical current is not always normalizable. For example, one can have many classical situations in which the event may never occur. The quantum case is more complicated however, since currents which are classically positive definate may be negative in the quantum case [29].

Lastly,

Property 4 *The probabilities $P_i(t)$ are positive definite.*

In general, $p_a(t)$ can be negative since $P(t)$ need not be monotonically increasing with time (this is obviously also true for classical probability functions). One can restrict $\mathbf{J}_a$

to only act on states for which $\mathbf{P}_a$ is increasing with time, but the restricted domain of definition of $\mathbf{J}_a$ may mean that it will no longer be self-adjoint. Furthermore, whether $\mathbf{J}_a$ is positive or negative will not only depend on the state, but also on the Hamiltonian. For certain Hamiltonians, one may find that there are no states for which $p_a(t)$ does not take on negative values.

Another interesting aspect of $\mathbf{J}_a$ and $\mathbf{\Pi}_a$ is that in general

$$[\mathbf{J}_a(t), \mathbf{\Pi}_a(t)] \neq 0. \tag{2.15}$$

The operator which measures that the event happened and the operator $\mathbf{J}_a$ do not commute. If one believes that $\mathbf{J}_a$ can be used to answer the question "when did the event happen?" then one finds that "when did it happen?" and "has it already happened?" seem to be complimentary (in Bohr's sense) in that they interfere with each other. Naively, it would seem that determining "when did a occur?" would also answer the question "has a occurred?". However the inaccuracy of the determination of "when did a occur?" seems to place limits on our ability to answer "has a occurred?".

2.3 Time of a Measurement or Arrival

We now examine two specific examples of the determination of when an event occurred. In the first example, one tries to determine when a measurement occurred. In the second example, one wishes to determine the time at which a particle arrives to $x = 0$

Let us try to measure the time that a measurement occurred (a measurement of a measurement in a sense). Imagine that we want to find out the time of a measurement of the observable $\mathbf{A}$ of a quantum system S. The measurement of $\mathbf{A}$ can be accomplished by coupling a macroscopic apparatus O to the system, via an interaction such as

$$H = g(t)\mathbf{PA} \tag{2.16}$$

where $\mathbf{P}$ is the conjugate momentum to the pointer $\mathbf{Q}$ of the measuring device, and $g(t)$ is a function which is zero everywhere, except during a small interval of time. After the measurement is complete, the measuring apparatus will be correlated with the state of the system. If initially, S is in a superposition of eigenstates $|\phi_i\rangle$ of the observable $\mathbf{A}$, so that $|\psi_S\rangle = \sum_i c_i|\phi_i\rangle$, then we expect the initial state of the combined $S - O$ system to evolve into a correlated state.

$$\sum_i c_i|\phi_i\rangle \otimes |O\rangle \to \sum_i c_i|\phi_i\rangle \otimes |O_i\rangle \tag{2.17}$$

where $|O\rangle$ is the original state of the device and the $|O_i\rangle$ are orthogonal states of the measuring apparatus which are correlated with the system. If the coupling is small, then the duration of the measurement might need to be long in order to distinguish between the various eigenvalues of $\mathbf{A}$. At any time during the measurement, it is possible to calculate the density matrix of the combined S-O system. One can imagine that a second apparatus O' measures the state of the first apparatus O to determine whether a measurement has occurred. This has been studied for the case when the measurement is gradual [21] [27]. Rovelli [28] has recently proposed that the apparatus O' might measure the operator

$$\mathbf{M} = \sum_i |\phi_i\rangle \otimes |O_i\rangle\langle O_i| \otimes \langle\phi_i|. \tag{2.18}$$

This is a projector onto the space of states where a correlation exists between the measuring apparatus and the quantum system. In the measurement scheme proposed by Rovelli, the probability that a measurement has been made at time t is given in the Heisenberg representation by

$$P_M(t) = \langle\psi_{SO}|\mathbf{M}(t)|\psi_{SO}\rangle \tag{2.19}$$

where $|\psi_{SO}\rangle$ is the state of the combined S-O system. The operator which is defined to give the probability that a measurement was made between times t and $t + dt$ is

$$\mathbf{m}(t) = \frac{d\mathbf{M}(t)}{dt}. \tag{2.20}$$

In the case of time-of-arrival, one wishes to measure the time a particle arrives to a certain location (say $x = 0$). Often, the probability current is used to determine the arrival time[13]. One imagines that a particle is localized in the region $x < 0$ and traveling towards the origin. The projector

$$\mathbf{\Pi}_+ = \int_0^\infty dx|x\rangle\langle x| \tag{2.21}$$

is an operator which is equal to one when $x > 0$ and zero otherwise. The probability of detecting the particle in the positive x-axis is given by

$$P_+(t) = \langle\psi|\mathbf{\Pi}_+(t)|\psi\rangle. \tag{2.22}$$

In the Schrödinger representation, this expression is just $P_+(t) = \int_0^\infty |\psi(x,t)|^2 dx$. It is then claimed that the current $\mathbf{J}_+$, given by

$$\frac{\partial \mathbf{J}_+}{\partial x} = \frac{d\mathbf{\Pi}_+(t)}{dt} \tag{2.23}$$

will give the probability that the particle arrives between t and $t + dt$.

It is clear that both the operators $\mathbf{M}(t)$ and $\mathbf{\Pi}_+(t)$ are specific example of the operator $\mathbf{\Pi}_a$ discussed in Section 2.2. $\mathbf{M}(t)$ gives the probability at time t that a measurement has occurred. $\mathbf{\Pi}_+(t)$ gives the probability that the particle is found at $x > 0$ at time t. The two operators $\mathbf{m}(t)$ and $\frac{\partial \mathbf{J}_+(t)}{\partial x}$ are examples of $\mathbf{J}_a(t)$. $\mathbf{m}(t)$ gives the change in the probability that the measurement happened at time t, while $\frac{\partial \mathbf{J}_+(t)}{\partial x}$ gives the change in the probability that the particle is found at $x > 0$. However, one cannot interpret these operators as giving the probability that the measurement occurred (or the particle arrived). None of these operators allow one to measure the precise time at which the event occurred. They do not posses all the Properties 1-4 listed above.

Considered as probabilities in time, none of the operators above give distributions which have Property 1. The operators above do not commute with the Hamiltonian, and therefore depend on t. For $t - t' \ll d\mathbf{H}$ we have for any operator $\mathbf{A}(t) \simeq \mathbf{A}(t') + i(t - t')[\mathbf{H}, \mathbf{A}(t')]$, and so

$$[\mathbf{A}(t), \mathbf{A}(t')] \simeq i(t - t')[\mathbf{H}, \mathbf{A}(t')], \mathbf{A}(t')] \tag{2.24}$$

For arbitrary Hamiltonians, it is obvious that none of the operators above will commute with themselves at different times. Even for a free particle, one can explicitly calculate that neither $\frac{\partial \mathbf{J}_+(t)}{\partial x}$ nor $\mathbf{\Pi}_+(t)$ commute with themselves at different times, (the calculation is neither difficult, nor particularly illuminating).

For some very specific states, and physical situations, Properties 2-4 may be obeyed, but this is certainly not true in general. For the case of time-of-arrival, even for a free Hamiltonian and wave packets which only contain modes of positive frequency, the current can be negative [29] (a violation of Property 4 - that probabilities must be positive definite). In fact, since the current is simply the time derivative of a projection operator, there is no reason to expect it to always be positive. For free particles which can arrive from the left and right, the current can be zero [2] and hence the probability distribution will be unnormalizable (Property 3). Also, in general, there is no reason why a particle can't be at the same position at many different times (a violation of Property 2). In the case of particles which move in a potential, one may find that there are no states for which Properties 3-4 are obeyed. For example, if there is an infinite potential barrier around the origin, the particle will never arrive, and the current will not be normalizable, and if there is a harmonic oscillator potential, the particle will cross the origin many times from both the left and the right violating Properties 2 and 4. For the case of determining when a measurement occurred, Rovelli restricts the class of measurements he considers to be those for which $\mathbf{m}(t)$ obeys Properties 2-4. As a result, $\mathbf{m}(t)$ cannot be used for arbitrary measurements. As with the time-of-arrival, there are clearly many Hamiltonians

[2] See Appendix A where we see that a coherent antisymmetric superposition of left and right moving waves has zero current.

for which Property 2-4 will be violated. Nor can $\mathbf{m}(t)$ be used for Hamiltonians for which its restricted domain of definition will mean that it is no longer self-adjoint.

Although operators such as $\mathbf{m}$ and and $\mathbf{J}_+$ do not commute with themselves at different times, it is possible to construct an operator which is time-translation invariant, and would give the time of an event in the classical limit. This will be discussed in Chapter 4 where we shall see that such an operator cannot be self-adjoint if the Hamiltonian is bounded from above or below.

2.4 Continual Event Monitoring

Instead of considering operators, a more physical meaningful method of measuring the occurrence of an event is to consider continuous measurement processes. For example, the operator $\mathbf{\Pi}_a(t)$ can be measured continuously or at small time intervals. When one considers such a physical measurement procedure one can see that the time at which an event occurs is not well defined in quantum mechanics. The probability of finding that the system enters one of the states ϕ_i at time t_a is given by the probability that it isn't in any of the states ϕ_i before t_a, times the probability that it is in one of the states ϕ_i at t_a.

To see how such a scheme might work, let us see how one would measure the time of an occurrence of the event corresponding to $\mathbf{\Pi}_a$. A measurement of the operator $\mathbf{\Pi}_a(t)$ will tell us whether the event has occurred at time t. We can then measure $\mathbf{\Pi}_a(t)$ at times $t_k = k\Delta$ for integral k in order to determine when the measurement occurred. Δ represents the frequency with which we monitor the system, and is therefore the inaccuracy of the measurement in time (it is the coarseness of the measurement in some sense).

We will now work in the Schrödinger representation, simply because it is the most natural arena to talk about successive measurements on a system. At time t_1, the probability that an event has occurred is given by

$$P(\uparrow, t_1) = \langle\psi_0(0)|\mathbf{U_\Delta}^\dagger \mathbf{\Pi}_a \mathbf{U_\Delta}|\psi_0(0)\rangle \tag{2.25}$$

and the probability that it hasn't is

$$P(\downarrow, t_1) = \langle\psi_0(0)|\mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}|\psi_0(0)\rangle \tag{2.26}$$

where $\uparrow$ corresponds to detecting that the event has occurred, $\downarrow$ corresponds to detecting that an event has not yet occurred, $\psi_0(0)$ is the initial state of the system and $\mathbf{U_\Delta}$ is the evolution operator $e^{-i\mathbf{H}\Delta}$. If the result is $\downarrow$, we collapse the wave function and evolve it to the next instant. The normalized state before the second measurement is:

$$|\psi_2(t_2)\rangle = \frac{\mathbf{U_\Delta}(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}|\psi_0\rangle}{\langle\psi_0|\mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}|\psi_0\rangle^{1/2}} \tag{2.27}$$

The probability that an event has occurred at t_2 is given by the probability that an event didn't occur at t_1 times the probability that ψ_2 is in one of the states ϕ_i

$$P(\uparrow, t_2) = \frac{\langle\psi_0|\mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}^\dagger \mathbf{\Pi}_a \mathbf{U_\Delta}(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}|\psi_0\rangle}{\langle\psi_0|\mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}|\psi_0\rangle} \times \langle\psi_0|\mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}|\psi_0\rangle \tag{2.28}$$

The probability that an event didn't occur is given by

$$P(\downarrow, t_2) = \langle\psi_0|\mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}|\psi_0\rangle \tag{2.29}$$

By repeating this process, we find that at time t_k the probability that an event has occurred is given by

$$P(\uparrow, t_k) = \langle\psi_0|A_k|\psi_0\rangle \tag{2.30}$$

where

$$A_k = \mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)...\mathbf{U_\Delta}^\dagger \mathbf{\Pi}_a \mathbf{U_\Delta}...(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta} \tag{2.31}$$

and the probability that an event hasn't occurred is

$$P(\downarrow, t_k) = \langle\psi_0|B_k|\psi_0\rangle \tag{2.32}$$

with

$$B_k = \mathbf{U_\Delta}^\dagger(\mathbf{1}-\mathbf{\Pi}_a)\mathbf{U_\Delta}^\dagger(\mathbf{1}-\mathbf{\Pi}_a)...\mathbf{U_\Delta}^\dagger(\mathbf{1}-\mathbf{\Pi}_a)\mathbf{U_\Delta}...(\mathbf{1}-\mathbf{\Pi}_a)\mathbf{U_\Delta}(\mathbf{1}-\mathbf{\Pi}_a)\mathbf{U_\Delta} \tag{2.33}$$

By allowing the unitary operators to act on the projection operators we can write the A_k or B_k in the Heisenberg representation. For example

$$A_k = (\mathbf{1}-\mathbf{\Pi}_a)(t_1)...(\mathbf{1}-\mathbf{\Pi}_a)(t_{k-1})\mathbf{\Pi}_a(t_k)(\mathbf{1}-\mathbf{\Pi}_a)(t_{k-1})...(\mathbf{1}-\mathbf{\Pi}_a)(t_1) \tag{2.34}$$

However, while the operators $\mathbf{\Pi}_a(t)$ can be found by unitary time-evolution of $\mathbf{\Pi}_a(0)$, the operators A_k and B_k are not related by a unitary transformation to A_0 and B_0. This already signals that they can not give an undisturbed distribution for the time of an event. Nor are the A_k and B_k projection operators.

The probabilities derived from A_k and B_k are not universal. In this case, they apply only to the specific measurement scenario under discussion. In particular the probability distribution is sensitive to the frequency at which $\mathbf{\Pi}_a$ is measured, a phenomenon which is related to the Zeno paradox [23].

As an example, consider a measurement of the spin of a particle. We wish to know at what time the measurement occurred. The particle is in a state given by

$$|\psi_S\rangle = a|\uparrow\rangle + b|\downarrow\rangle \tag{2.35}$$

and we use a simple measuring device which is also a spin 1/2 particle initially in the state $|O\rangle = |\uparrow'\rangle$, which evolves according to the Hamiltonian

$$\mathbf{H} = g(t)\sigma'_x\frac{1}{2}(1-\sigma_z) \tag{2.36}$$

where $\int g(t)dt = \pi$ ($g(t)$ is sharply peaked, with width T), and the primed Pauli matrix acts on the measuring device, while the unprimed Pauli matrix acts on the system. After a time T, the spin of the measuring device will be correlated with the system. Since this measurement is rather crude, (the initial state of the device is the same as one of the measurement states), the operator $\mathbf{M}$ at $t = 0$ is not zero. Let us simplify the problem further, by assuming that $a = 0$ and $b = 1$. In this case, the only relevant matrix element of $\mathbf{1} - \mathbf{M}$ is $|\downarrow\rangle \otimes |\uparrow'\rangle\langle\uparrow'| \otimes \langle\downarrow| = |\psi_o\rangle\langle\psi_o|$. We then find the probability that the measuring apparatus has not responded at time t_k is

$$\begin{aligned} P(\downarrow, t_k) &= |\langle\psi_o|U_\Delta|\psi_o\rangle|^{2k} \\ &\simeq |\langle\psi_o|1 - i\Delta\mathbf{H} - \Delta^2\mathbf{H}^2|\psi_o\rangle|^{2k} \\ &\simeq 1 - \Delta^{2k}(\langle\mathbf{H}^2\rangle - \langle\mathbf{H}\rangle^2)^k \end{aligned} \tag{2.37}$$

If we fix a value of $\tau = t_k$ and then make Δ go to zero, we find

$$\begin{aligned} P(\downarrow, \tau) &\simeq e^{-(\Delta dE)^2\tau/\Delta} \\ &\simeq 1, \end{aligned} \tag{2.38}$$

which implies that the measuring apparatus becomes frozen and never records a measurement. In order not to freeze the apparatus, we need $\Delta > 1/dE$ where dE is the uncertainty in energy of the measuring device O (initially the spacing between energy levels in this case). There is always an inherent inaccuracy when measuring the time that the event (of the measurement) occurred. This inaccuracy is similar to the one which we will find in Chapter 3. Note that as discussed in the Introduction, this inaccuracy is not related to the so-called "Heisenberg energy-time uncertainty relationship" as it applies to every single measurement and not to the width of measurements carried out on an ensemble.

One can of course use the set of operators A_k to compute a probability distribution in time, or experimentally determine a probability distribution for the time of an event. However, as we have just seen, this probability distribution is not a function of the system alone, but rather, it is related to the system and the measuring device (or set of operators) For example, the probability distribution will depend on Δ, and if Δ is too small, we will find that the event never occurs. The distribution $P(\uparrow, t_k)$ does allow as to predict the probabilities of future measurements using a particular measuring device, but the results are not attributes of the system.