Appendix A

Zero-Current Wavefunctions

One interesting aspect of the detector discussed in Section 3.3.2, is that while it can be used for wave-packets arriving from the left or the right, it will not always be triggered if the wavefunction is a coherent superposition of right and left moving modes. Consider for example, the superposition

$$\psi(x) = Ae^{ikx} + Ae^{-ikx}. \tag{A.216}$$

One can easily verify that the current

$$j(x,t) = -i\frac{1}{2m}\left[\psi^*(x,t)\frac{\partial\psi(x,t)}{\partial x} - \frac{\partial\psi^*(x,t)}{\partial x}\psi(x,t)\right] \tag{A.217}$$

is zero in this case. $|\psi(0,t)|^2$ is non-zero, although the state is not normalizable. As in eq. (3.63) this state evolves into

$$\langle x|\psi\rangle|\uparrow_z\rangle \quad \rightarrow \quad \frac{A}{\sqrt{2}}\Big[(e^{ikx} + e^{-ikx})|\uparrow_x\rangle + (e^{ikx} + e^{-ikx})|\downarrow_x\rangle\Big] \tag{A.218}$$

Which, when rewritten in the σ_z basis, is just

$$A(e^{ikx} + e^{-ikx})|\uparrow_z\rangle. \tag{A.219}$$

i.e. the detector is never triggered.

This wavefuntion is similar to the antisymmetric wavefunctions discussed by Yamada and Takagi in the context of decoherent histories [36] and Leavens [37] in the context of Bohmian mechanics, where also one finds that the particles never arrive. How to best treat these cases is an interesting open question.

Appendix B

Gaussian Wave Packet and Clocks

Using the simple model of Section 3.3.1 (3.44), we now calculate the probability distribution of a clock which measures the time-of-arrival of a Gaussian wave packet. We will perform the calculation in the limits when the clock is extremely accurate and extremely inaccurate. The wave function of the clock and particle is given by (3.52) and the distributions are both Gaussians given by (3.53). In the inaccurate limit, when $p_o << k$, $A_T \sim 1$. We trace over the position of the particle on the condition that the clock was triggered, ie. $x > 0$.

$$\begin{aligned} \rho(y,y)_{x>0} &= \int dx |\psi(x>0,y,t)|^2 \\ &\simeq N^2 \int_{-\infty}^{\infty} dk\, dk' \int_0^{\infty} dp\, dp'\, dx g(k) g^*(k') f(p) f^*(p') e^{i(q-q')x + i(p-p')y - \frac{i(q^2-q'^2)t}{2m}} \end{aligned} \tag{B.220}$$

After a sufficiently long time, ie. $t >> t_a$ the wave function has no support on the negative x-axis, and if $p_o > 1/\Delta y$, then it will not have support in negative p. We can thus integrate p and x over the entire axis. Integrating over x gives a delta-function in q. We can then integrate over p' to give

$$\rho(y,y)_{x>0} \simeq \frac{2\pi N^2}{m} \int dk\, dk'\, dp \sqrt{k^2 + 2mp}\, g(k) g^*(k') f(p) f^*(p + \frac{k^2 - k'^2}{2m}) e^{i(k'^2 - k^2)\frac{y}{2m}}$$

where we have used the fact that $\delta(f(z)) = \frac{\delta(z-z_o)}{f'(z=z_o)}$ when $f(z_o) = 0$. The square root term varies little in comparison with the exponential terms and can be replaced by its average value $\sqrt{k_o^2 + 2mp_o} \simeq k_o$. Integrating over p gives

$$\rho(y,y)_{x>0} \simeq \frac{2\pi N^2 k_o}{m} \sqrt{\frac{\pi}{2\Delta y^2}} \int dk\, dk'\, e^{\frac{-\Delta y^2}{8m^2}(k+k')^2(k-k')^2} g(k) g^*(k') e^{i(k'^2-k^2)\frac{y}{2m}}. \tag{B.221}$$

Since $\Delta y\, k >> 1$, for a wave packet peaked around k_o we can approximate the argument of the first exponential by $\frac{-\Delta y^2 k_o^2}{2m^2}(k-k')^2$. This allows us to integrate over k and k'

$$\rho(y,y)_{>0} \simeq \frac{1}{\sqrt{2\pi\gamma(y)}} e^{-\frac{(y-t_c)^2}{2\gamma(y)}} \tag{B.222}$$

where the width is $\gamma(y) = \Delta y^2 + (\frac{m\Delta x}{k_o})^2 + (\frac{y}{2k_o\Delta x})^2$.

As expected, the distribution is centered around the classical time-of-arrival $t_c = x_o m/k_o$. The spread in y has a term due to the initial width Δy in clock position y. The second and third term in $\gamma(y)$ is due to the kinematic spread in the time-of-arrival $1/dE = \frac{m}{kdk}$ and is given by $\frac{dx(y)\,m}{k_o}$ where $dx(y)^2 = \Delta x^2 + (\frac{y}{2m\Delta x})^2$. The y dependence in the width in x arises because the wave packet is spreading as time increases, so that at later y, the wave packet is wider. As a result, the distribution differs slightly from a Gaussian although this effect is suppressed for particles with larger mass.

When the clock is extremely accurate ie. $p_o >> k_o$ we have $A_T \sim k\sqrt{\frac{2}{mp}}$.

$$\begin{aligned}
\rho(y,y)_{x>0} &\simeq \frac{2N^2}{m}\int_{-\infty}^{\infty} dk\, dk' \int_0^{\infty} dp\, dp'\, dx \frac{kk'}{\sqrt{pp'}} g(k)g^*(k')f(p)f^*(p')e^{i(q-q')x+i(p-p')y-\frac{i(q^2-q'^2)t}{2m}} \\
&\simeq \frac{4\pi N^2}{m}\int dk\, dk'\, dp \frac{kk'}{m}\sqrt{\frac{k^2+2mp}{p(p+\frac{k^2-k'^2}{2m})}} g(k)g^*(k')f(p)f^*(p+\frac{k^2-k'^2}{2m})e^{i(k'^2-k^2)\frac{y}{2m}}
\end{aligned}$$

Since $p_o >> k_o$, we can approximate this integral as

$$\rho(y,y)_{x>0} \simeq \frac{A}{m} \,|\int dk\, k\, g(k)e^{-i\frac{k^2 y}{2m}}|^2 \tag{B.223}$$

where $A \equiv 4\pi\sqrt{\frac{2}{m}}N^2\int\frac{dp}{\sqrt{p}}|f(p)|^2$. We can approximate p by p_o to take it outside the integrand, giving

$$A \simeq \sqrt{\frac{\pi}{mp_o}}\frac{2\Delta x}{\pi^2}. \tag{B.224}$$

The final integration over k yields

$$\rho(y,y)_{>0} \simeq 4\sqrt{\frac{k_o^2}{2mp_o}\frac{\tilde{\gamma}(t_c)}{\tilde{\gamma}(y)}}\frac{1}{\sqrt{2\pi\tilde{\gamma}(y)}}e^{-\frac{(y-t_c)^2}{2\tilde{\gamma}(y)}} \tag{B.225}$$

where the width $\tilde{\gamma}(y) = \Delta x^2 + (\frac{y}{2k_o \Delta x})^2$ is independent of Δy because the kinematic spread in the time-of-arrival $1/dE$ is much larger than the spread in the position of the clock. In this limit we see two additional factors. The amplitude decays like $\sqrt{E_o/p_o}$ so that improved accuracy decreases our chances of detecting the particle. Also, there is a minor correction of $\frac{\tilde{\gamma}(t_c)}{\tilde{\gamma}(y)}$. More energetic particles with faster arrival times are more likely to trigger the clock.

Appendix C

Time-of-Arrival Eigenstates

We will now show that the eigenstates of [9] and those of the unmodified time of arrival operator do not correspond to a delta function at the time-of-arrival t_A but are instead proportional to $x^{-\frac{3}{2}}$. Using the Schrödinger representation, we see that at time t_A, these eigenstates (eg ${}_\epsilon\tilde{g}^+_{t_A}(x, t = t_A)$ of eqn. (4.4)) in the x-representation are given by

$$\begin{aligned}
\langle x|T^+_\epsilon\rangle &= \int_{-\infty}^{\infty} dk e^{-ikx} e^{-it_A \frac{k^2}{2m}} {}_\epsilon g^+_{t_A}(k) \\
&= \int_0^\epsilon dk e^{-ikx} e^{-it_A \frac{k^2}{2m}} \sqrt{\frac{\hbar}{2\pi m}} \frac{\epsilon}{\sqrt{k}} \exp\left(i\frac{\hbar T}{m}\int_{+\epsilon}^k dk' \frac{\epsilon^2}{k}\right) + \\
&\quad \int_\epsilon^\infty dk e^{-ikx} e^{-it_A \frac{k^2}{2m}} \sqrt{\frac{\hbar}{2\pi m}} \sqrt{k} \exp\left(\frac{i\hbar t_A}{2m}(k^2 - \epsilon^2)\right) \\
&= \epsilon\sqrt{\frac{\hbar}{2\pi m}} \int_0^\epsilon dk e^{-ikx} e^{-it_A \frac{k^2}{2m}} \frac{1}{\sqrt{k}} \exp\left(i\epsilon^2 \frac{\hbar t_A}{m}(\ln(k) - \ln(\epsilon))\right) + \\
&\quad \sqrt{\frac{\hbar}{2\pi m}} \exp(-\frac{i\hbar t_A}{2m}\epsilon^2) \int_\epsilon^\infty dk e^{-ikx} \sqrt{k}
\end{aligned} \tag{C.226}$$

As ϵ goes to zero the first integral goes to zero. So we get

$${}_\epsilon g^+_{t_A}(x, t = t_A) = \sqrt{\frac{\hbar}{2\pi m}} x^{-3/2} \frac{\sqrt{\pi}}{2} \tag{C.227}$$

where we have added a small imaginary part to x to make the integral converge, and then set it to zero at the end. One finds the same behaviour for the eigenstates of the unregularized time-of-arrival operator $|T\rangle$. The reason is that as ϵ goes to zero, the modification of the eigenstates only occurs at $k = 0$ which is a set of measure zero.